Relating Foliations and Dynamical Systems

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Introduction

Dynamical Systems

Given a topological space X, a *dynamical system* is a triple $(X, \mathbb{R}, \pi)$, where π is a continuous map $\pi : \mathbb{R} \times X \to X$, such that, for all $x \in X$, and $t_1, t_2 \in \mathbb{R}$,

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- $\pi(t_1, \pi(t_2, x)) = \pi(t_1 + t_2, x)$.

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We may use $tx = \pi(t, x)$ for brevity.
For purposes of this talk, we will use $\mathbb{R}^2$ or $\mathbb{R}^3$ as our phase space.

Definitions

For a given point $x \in X$,

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- *orbit* - $\gamma(x) = \{tx : t \in \mathbb{R}\}$.
- We also refer to the orbit as the *trajectory*.
- If there are no fixed points (where $tx = x$ for all $t \in \mathbb{R}$), then the dynamical system is *non-singular*.

Ways to describe a dynamical system

- Parametric equations

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- Potential function

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Parametric Example

Suppose we have the dynamical system from the function

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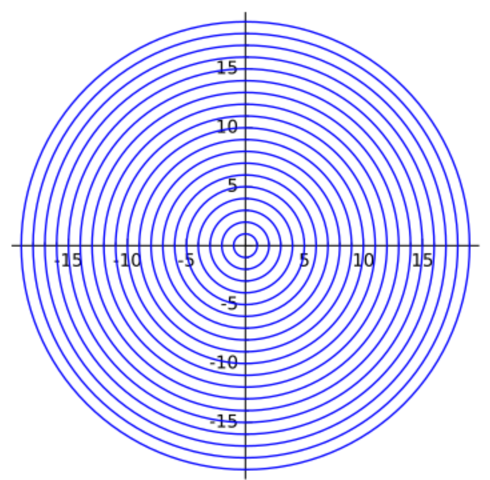
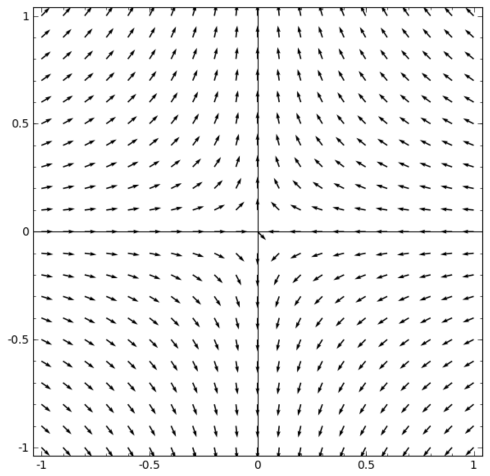


Vector Field Example

$$\dot{x} = -x \text{ and } \dot{y} = y$$



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In polar coordinates, consider the potential function

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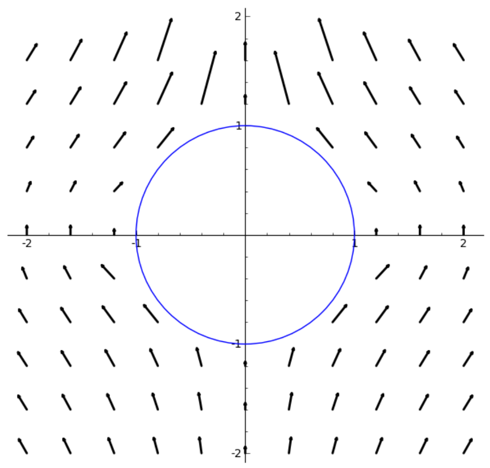

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- Call our function $\mu : X \to \mathbb{R}$, that takes in a subset of X, and returns a real number for the volume.
- If for any $t \in \mathbb{R}$, $\mu(A) = \mu(tA)$, then our dynamical system *volume-preserving*.

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$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\varphi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$$

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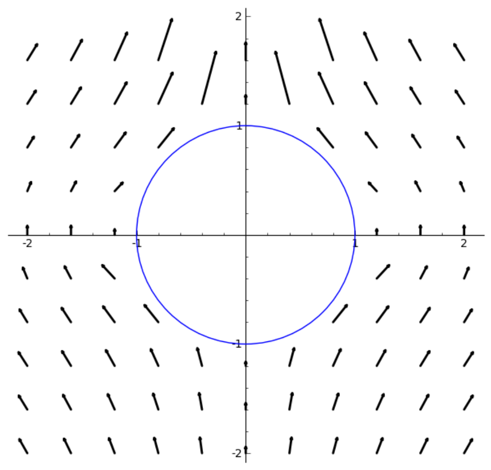
- It can be shown that this is equivalent to preserving volume, or in this 2-dimensional case, preserving area.

Example

Our flow does satisfy this condition, and hence, it preserves area.



Piecewise-linear dynamics

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- Subdivide into simplices.
- In the end, you have a triangulation.

Examples

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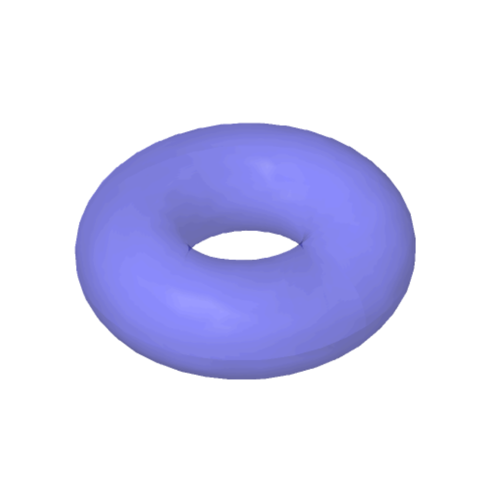

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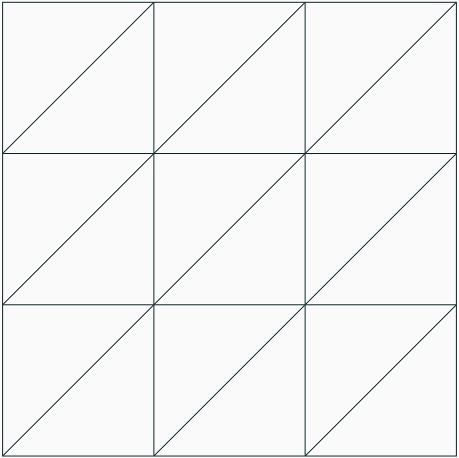

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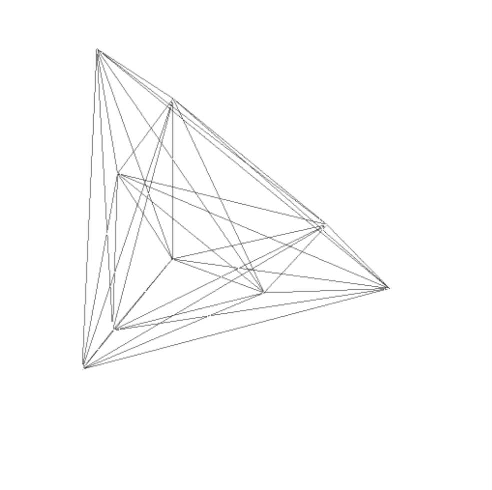

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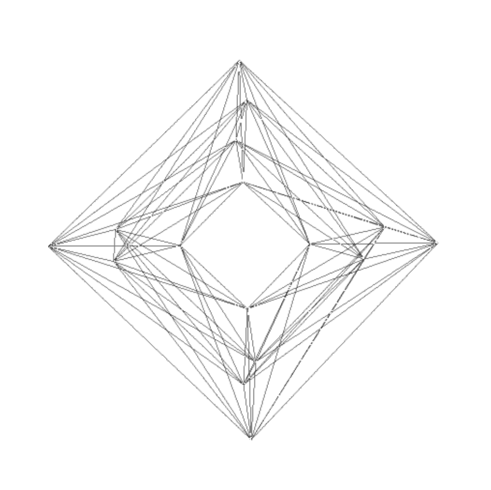

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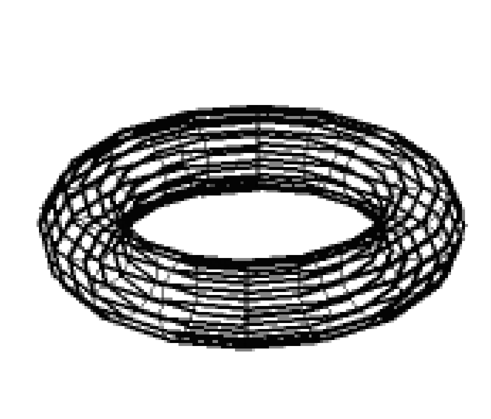

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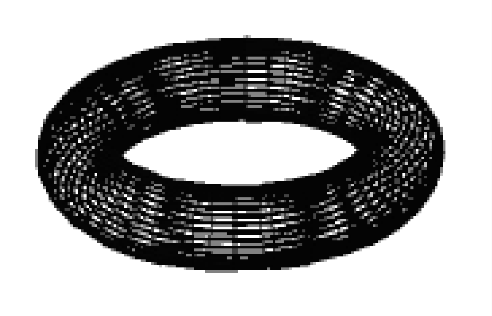

Piecewise-linear Dynamics

Given a phase space X, and a dynamical system $(X, \mathbb{R}, \pi)$, the system is *piecewise-linear* if the trajectories are linear on each simplex in the triangulation of X.

PL dynamics examples

Consider a 4-fold approximation of an annulus, with all trajectories spiraling in towards the center of the annulus.

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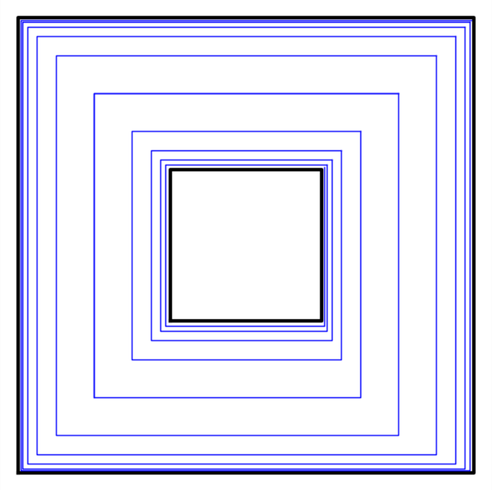

Foliations

1-foliations

Let S be an *atlas* on a phase space X, that is, a collection of open sets and maps, (U_i, φ_i), where $\varphi_i : U_i \to \mathbb{R}^n$ is a smooth embedding, and the set of all U_i covers M, with φ_i and φ_j agreeing on their overlap.

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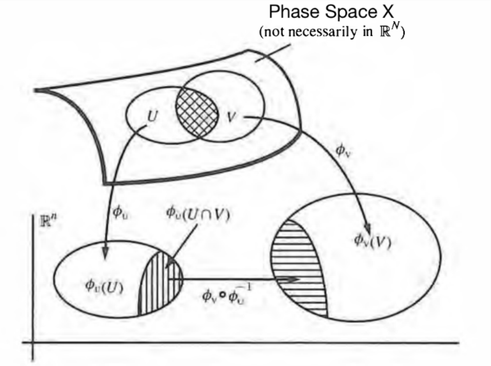

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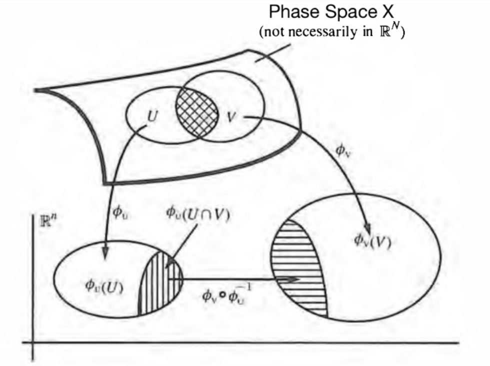

Each pair $(U_i, \varphi_i) \in S$ is known as a *chart*.

1-foliations

Fix some indexing set A. Let $\mathcal{F} = \{L_\alpha : \alpha \in A\}$ be a collection of arcwise connected subsets of M. $\mathcal{F}$ is a *1-dimensional folation of M* if

(i) $L_\alpha \cap L_\beta = \emptyset$ for $\alpha \neq \beta$

(ii) $\bigcup_{\alpha \in A} L_\alpha = M$.

(iii) Given any point $p \in M$, there exists a chart of $(U_\lambda, \varphi_\lambda)$ about p, such that for L_α with $L_\alpha \cap U_\lambda \neq \emptyset$, each path component of $\varphi(L_\alpha \cap U_\lambda)$ is of the form

$$\{x_1 \in \varphi_\lambda(U_\lambda) : x_2 = c_1, x_3 = c_2, \ldots, x_n = c_{n-1}\}$$

where each c_i is a constant determined by L_α.

Each L_α is a *leaf* of the foliation $\mathcal{F}$. We can view the embeddings as splitting $\mathbb{R}^n$ into two pieces, $\mathbb{R}$ and $\mathbb{R}^{n-1}$. On $\mathbb{R}$, the coordinates of the embedding vary with L_α, but on $\mathbb{R}^{n-1}$, the coordinates are fixed.

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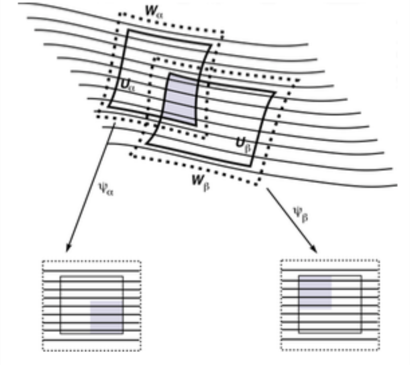

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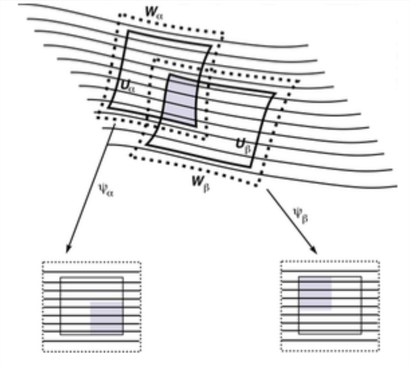
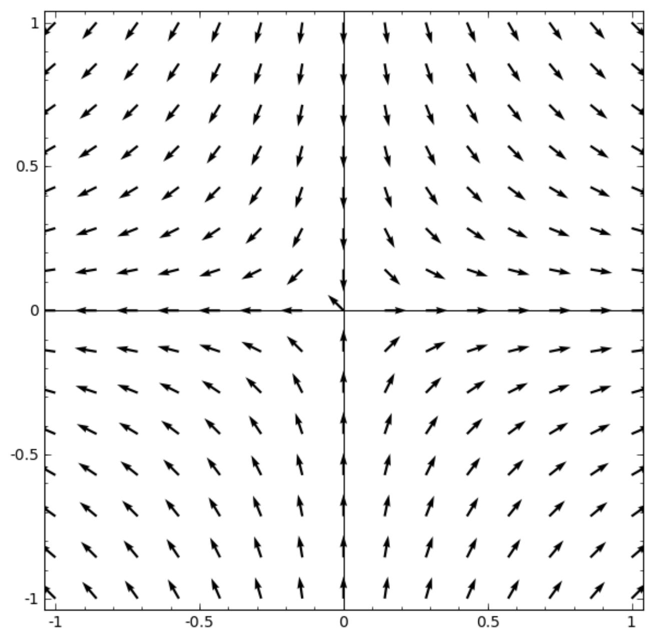


We say the foliation is *oriented* if we choose a direction in which we can move on the leaves.

Examples



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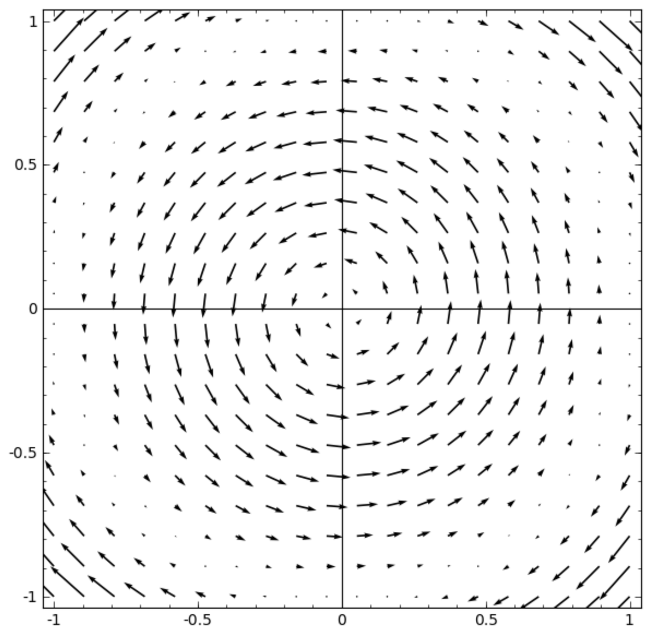

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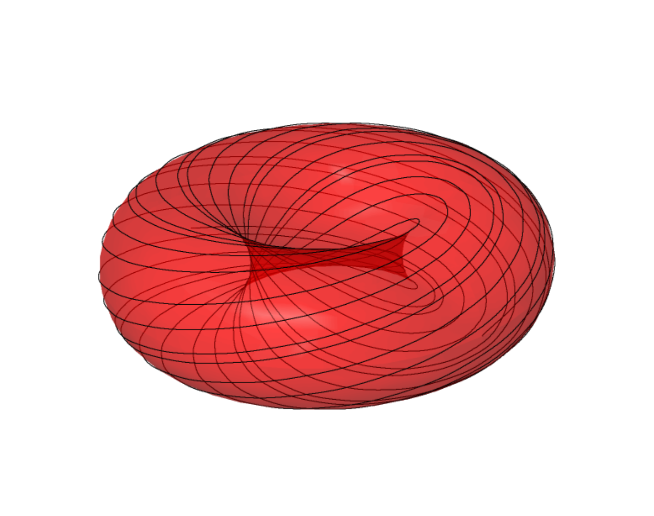

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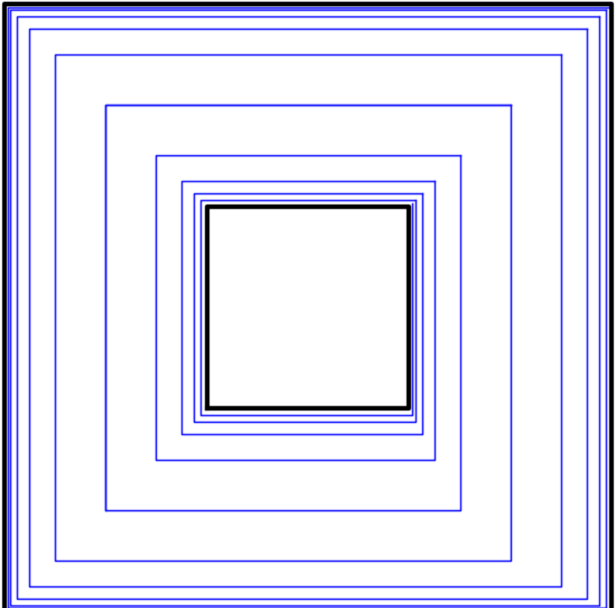

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- On each box a subset of X which is not parallel to any leaf, is a *small transversal*.

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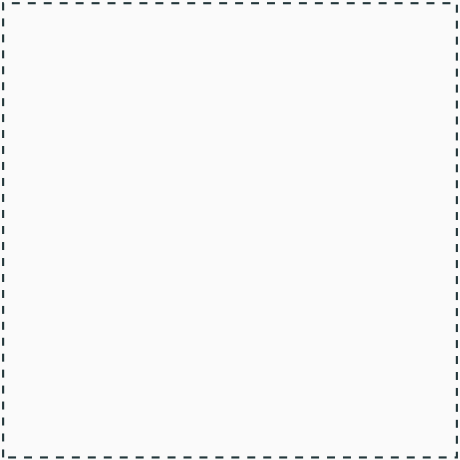
- On each box a subset of X which is not parallel to any leaf, is a *small transversal*.
- We need a function η which assigns a real number to each small transversal.

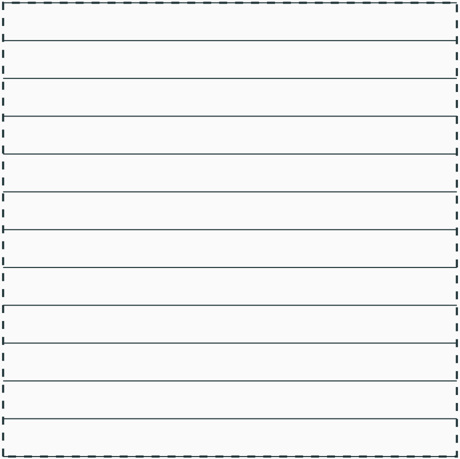
Examples

Start with a box



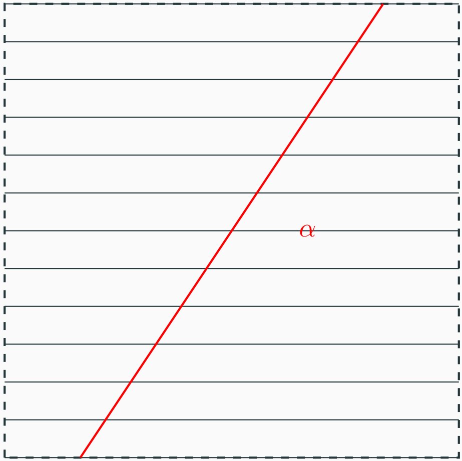
Examples

Only one coordinate changes on each leaf.



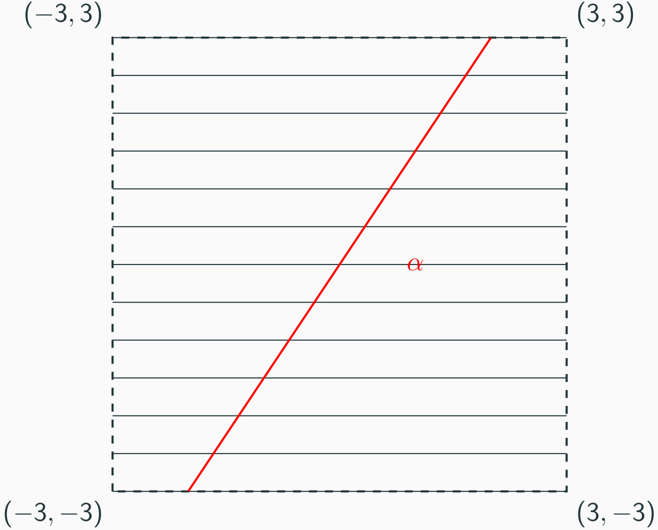
Examples

Add in a transversal, which we call α.



Examples

Throw in some coordinates, and I declare $\eta(\alpha) = 6$



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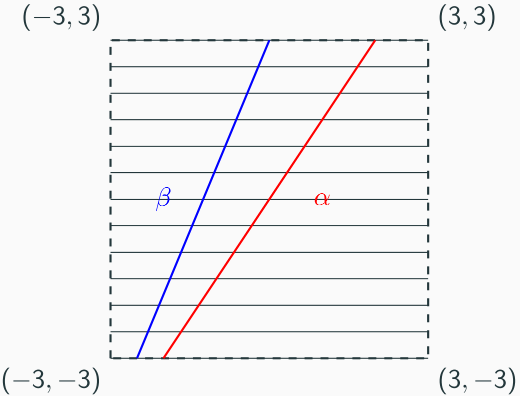

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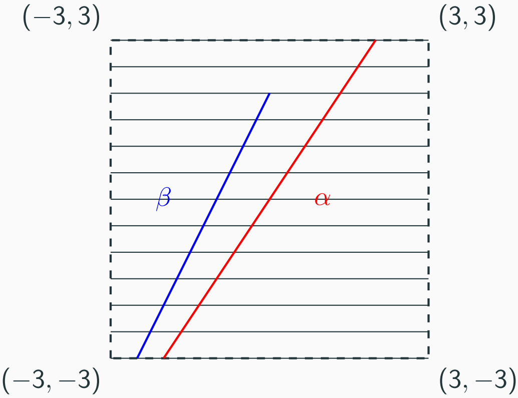


Given a space X, and a foliation $\mathcal{F}$ on X, we say that $\mathcal{F}$ is a *measured-foliation* with *measure* η, if, for any two isotopic small transversals α and β,

$$\eta(\alpha) = \eta(\beta).$$

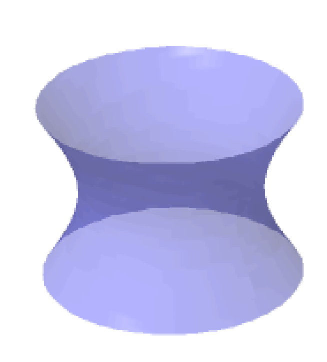
Relating foliations and dynamics

Volume-preserving dynamics $\Rightarrow$ measured-foliation

Let's take a hyperboloid of one sheet X, with parametric equations

- $x(u,v) = \sqrt{u^2+1}\cos(v)$
- $y(u,v) = \sqrt{u^2+1}\sin(v)$
- $z(u,v) = u$

for $u \in (-1,1)$ and $v \in [0,2\pi)$



For each $(x, y, z) \in X$, define

$$\pi(t,(x,y,z)) = \left(\sqrt{z^2+1}\cos(\tan^{-1} y/x + t), \sqrt{z^2+1}\sin(\tan^{-1} y/x + t), z\right)$$

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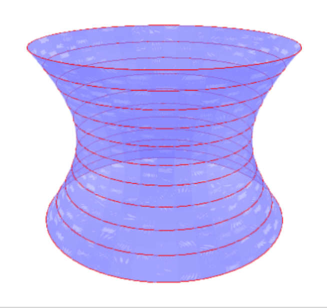
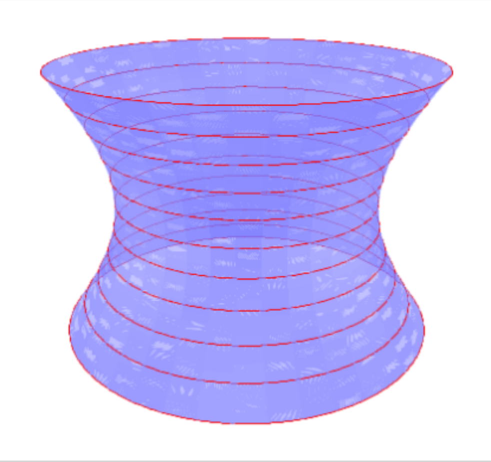
We can check that this is a dynamical system (by being careful with the arctangent, and that the trajectories on X look like this.



We can also check that this dynamical system is volume-preserving. Not too tough.



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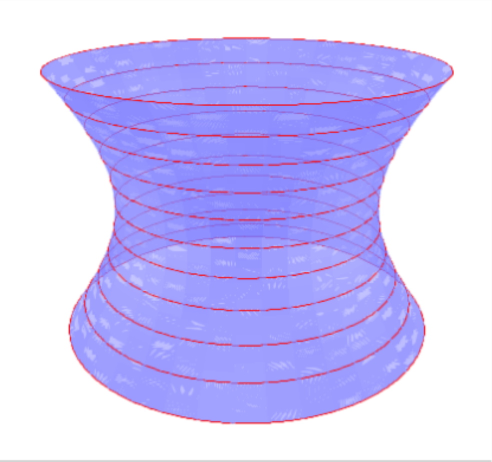

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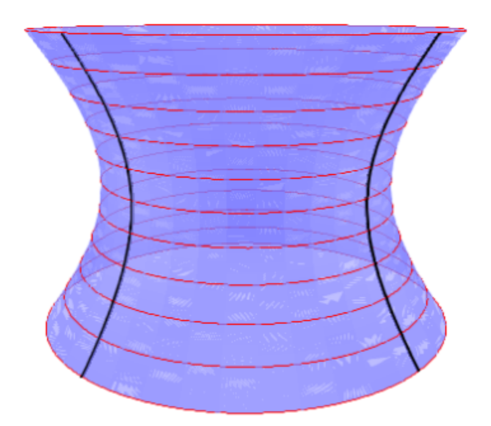

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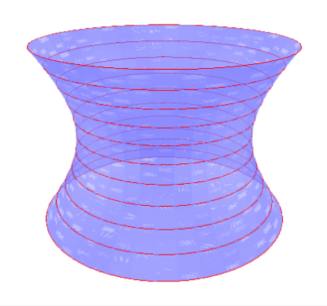
Measured foliations $\Rightarrow$ volume-preserving dynamics

What about the other direction?

Measured foliations ⇒ volume-preserving dynamics

What about the other direction?

- Start with a measured foliation.
- The change in all but one coordinate is preserved when we move between isotopic transversals.
- For each leaf, calculate it's length.
- Adjust the speed of the dynamical system so that the change in the remaining coordinate is preserved.



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Good luck building that directly from a dynamical system!
It's still not exactly easy with a foliation, but at least it's possible!