A volume-preserving dynamical system in $\mathbb{R}^3$ with bounded trajectories

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Abstract

In 1969, Jones and Yorke [3] produced a dynamical system on $\mathbb{R}^3$ with all trajectories bounded. This was accomplished using a countable set of nested tori of increasing size. We show here how, using a different collection of nested shapes, we can construct a dynamical system with all trajectories bounded, which also preserves volume.

Considerations

- All dynamical systems here are continuous. They are $\mathbb{R}$ actions on $\mathbb{R}^3$, generated by a vector field. The phase space is Ω, with map $\pi : \mathbb{R} \times \Omega \to \Omega$.
- If μ is a measure on Ω, such that for any measurable set $A \subset \Omega$ and any $t \in \mathbb{R}$, $\mu(A) = \mu(\pi(-t, A))$, then $(\mathbb{R}, \Omega, \pi, \mu)$ is a *measure-preserving* dynamical system [5].
- In the case that μ given by a smooth volume form and the dynamical system is parallel to a smooth vector field $\vec{v}$ on Ω, the measure-preserving condition is equivalent to the divergence equation [2]

$$\nabla \cdot \vec{v} = 0.$$

- Such a dynamical system is called *volume-preserving*

Jones-Yorke Construction [3]

- Define the function $c(r) = \frac{2}{3}(4^{r+1} - 4)$.
- Construct a set of tori, $\{T_r : r \in 0, 1, 2, \ldots\}$, where, for $u, v \in [0, 2\pi]$, T_r is the region bounded by the parametric surface

$$x = 4^r(2 + \cos(u))\cos(v)$$
$$y = 4^r(2 + \cos(u))\sin(v) + c(r)$$
$$z = 4^r \sin(u)$$

if r is even

$$x = 4^r \sin(u)$$
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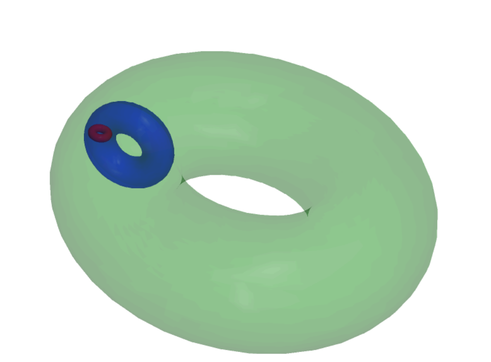
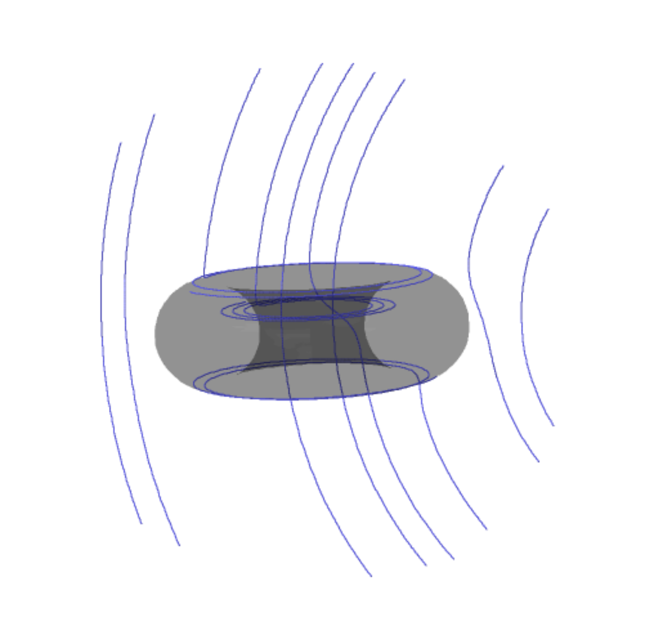


In [3], a flow on $\mathbb{R}^3$ is constructed which is non-singular, and with all trajectories bounded.

- Let $p = (x, y, z)$ be a point in $\mathbb{R}^3$.
- Let $i(r) = \frac{(-1)^{n+1}+1}{2}$
- Let $\mathfrak{G}_0(p) = (y, -x, 0)$
- $\mathfrak{G}_1(p) = (0, -z, y)$.
- $\mathfrak{h}_0(p) =$max$\{0$,min$\{1, 1 - d(T^0, p)\}\}$
- $\mathfrak{h}_r(p) =$max$\{0$,min$\{1, 1 - d(T^r, p), d(T^{r-1}, p)\}\}$, where d is the distance under the Euclidean metric.
- The flow on $\mathbb{R}^3$ is then

$$\vec{F}(p) = \sum_{r=0}^{\infty} \mathfrak{G}_{i(r)}(p - (0, c(r), 0))\mathfrak{h}_r(p). \tag{1}$$

Figure 2: Jones-Yorke flow local to T_0



Main result

Theorem

There exists a C^∞ non-singular volume-preserving dynamical system on $\mathbb{R}^3$, with all trajectories bounded.

Finding a volume-preserving flow around a torus

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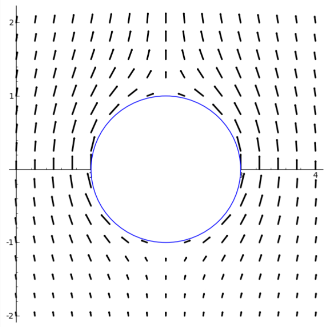

Figure 3: Divergence-free flow around a circle

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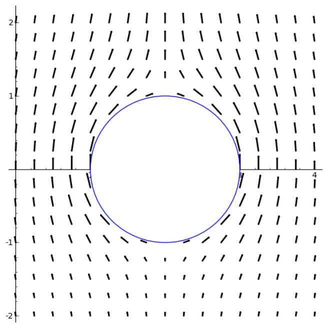

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Rotating this around the vertical axis yields a flow around a solid torus.

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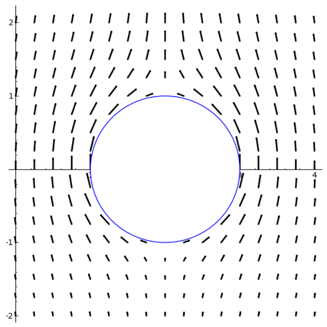

Figure 3: Divergence-free flow around a circle

Rotating this around the vertical axis yields a flow around a solid torus. This won't agree with the Jones-Yorke flow on the boundary of the torus.

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- We can use bump functions to make the flow agree with the Jones-Yorke flow on the boundary of the torus.
- Bump functions tend to mess up the divergence.
- Use cylindrical coordinates, with a bump function that only depends on r and z.

- Define a function which measures the distance of a point (r, θ, z) from the boundary of a torus T_n (when T_n is centered at the origin) by,

$$h_n(r,z) = \sqrt{z^2 + (r - 2 \cdot 4^n)^2} - 4^n$$

- Define the bump function

$$b(h) = \begin{cases} 1 & \text{if} \quad h \leq 0 \\ e^{-\frac{1}{1-(h-1)^2}+1} & \text{if} \quad h \in (0,1) \\ 0 & \quad \text{otherwise} \end{cases}$$

The flow is then given by

$$W_{T_n} = \langle 0, -b(h_n(r,z)) - r\theta \frac{\partial b(h_n(r,z))}{\partial z}, (b(h_n(r,z)) - 1).$$

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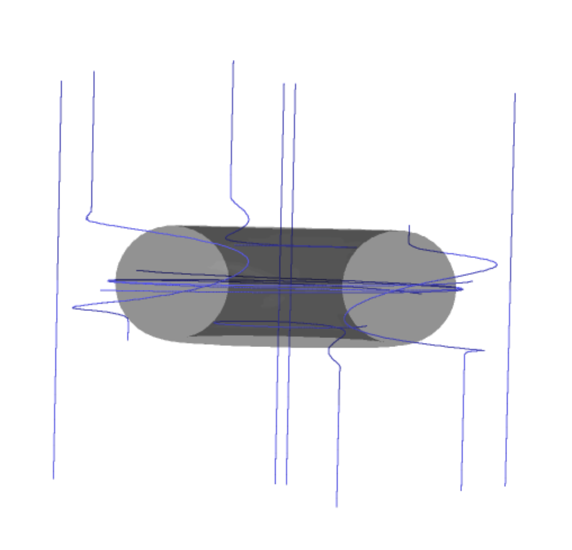

$$W_{T_n} = \langle 0, b_n + \frac{\partial b_n}{\partial z} r\theta(6 \cdot 2^{2n-1}) - \frac{\partial b_n}{\partial z} r^2 \cos(\theta), (1 - b_n)(r \sin(\theta) - 6 \cdot 2^{2n-1}) \rangle.$$

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- W_{T_n} is vertical at distance 1 from the boundary of T_n.
- This vertical component makes it difficult to embed this into the next torus in the Jones-Yorke construction.

Move away from the torus

Construct a new shape, which is diffeomorphic to a torus, and can address the issue with the vertical component.

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Construct a new shape, which is diffeomorphic to a torus, and can address the issue with the vertical component.
Define an *obround* of radius R as the set of points in $\mathbb{R}^2$

$$\{(x, y) \in \mathbb{R}^2 : x \in [-R, R], y = \pm R\} \cup \{(x, y) \in \mathbb{R}^2 : (x \pm 2R)^2 + y^2 = R^2\}.$$

An obround is the boundary of a square of side length $2r$, with semi circles of radius r appended to the right and left sides.

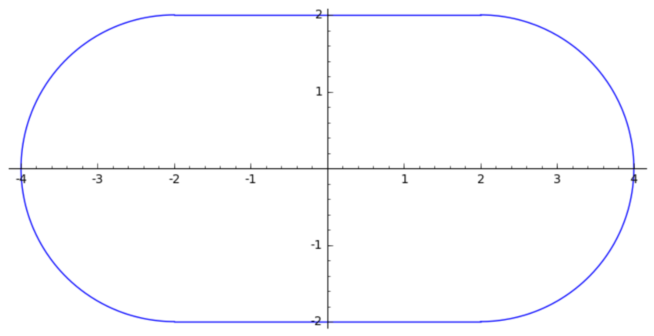

Figure 5: Obround with radius 2

A *tobround* of major radius R and minor radius r, with $r < R$, is the Cartesian product of a solid disk of radius r and an obround of radius R.

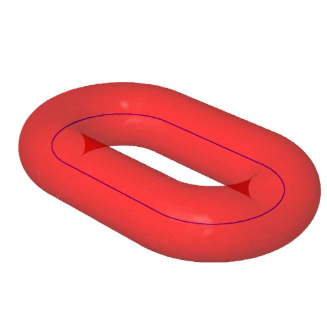

Figure 6: Tobround with major radius 2, and minor radius 1

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- Construct a smooth, volume-preserving flow around the torus, which is vertical at distance 1 from the boundary of the torus.
- Use the flow around the torus to construct a flow in a neighborhood of each tobround.
- The resulting flow smooth agrees with the existing flow on the nested tobrounds, is non-singular, has all trajectories trapped within a particular tobround, and preserves volume.

Define a sequence of tobrounds whose union is $\mathbb{R}^3$.

- Let $\mathcal{O}_0$ be a tobround with minor radius 1 and major radius 2
- If $n > 0$ is even, let $\mathcal{O}_n$ be the tobround with minor radius $6 \cdot 2^{2n-1}$ and major radius $6 \cdot 2^{2n}$, shifted $6 \cdot 2^{2n}$ units positively along the y-axis.
- If $n > 0$ is odd, the major and minor radii are as defined above, as is the shifting along the y-axis, but the entire obround is rotated about the $y-$axis by an angle of $\pi/2$.

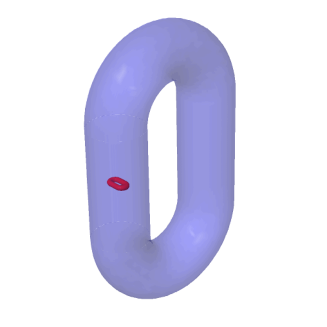

Figure 7: $\mathcal{O}_0$ nested in $\mathcal{O}_1$

Note that

$$\bigcup_{n \in \mathbb{N}} \mathcal{O}_n = \mathbb{R}^3.$$

Flow on nested tobrounds

- Construct a flow on $\mathbb{R}^3$ by defining it on each tobround.
- For each $p \in \mathbb{R}^3$, let $d(p, \mathcal{O}_n)$ be the usual distance function.
- For each $\mathcal{O}_n$, let $o_n(d(p, \mathcal{O}_n)$ be the smooth bump function which returns 1 on the boundary of $\mathcal{O}_n$ and 0 for all points whose distance from $\mathcal{O}_n$ is greater than or equal to 1.

Define a piecewise vector field in Cartesian coordinates on $\mathcal{O}_0$ by

$$\dot{p} = \begin{cases} \langle y, 0, 0 \rangle & \text{if } x \in [-2, 2] \\ \langle y, 2 - x, 0 \rangle & \text{if } x > 2 \text{ and } (x-2)^2 + y^2 \in [1, 9] \\ \langle y, -x - 2, 0 \rangle & \text{if } x < -2 \text{ and } (x+2)^2 + y^2 \in [1, 9] \end{cases}$$

The trajectories are then clockwise oriented obrounds, within the tobround $\mathcal{O}_I$. This is non-singular, and all trajectories are bounded.

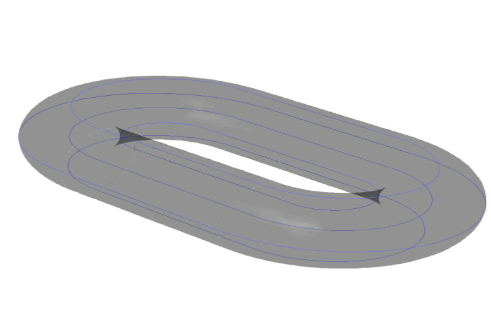

Figure 8: $\mathcal{O}_0$ with obround trajectories

Let o_0 be the bump function above, with argument assumed to be $d(p, \mathcal{O}_0)$. Extend our flow to $\mathcal{O}_1$ by

$$\dot{p} = \begin{cases} \langle o_0 y, 0, (1 - o_0)(y - 24)\rangle \\ \qquad\qquad\qquad \text{if } z \in [-24, 24] \\ \langle o_0 y, o_0(2 - x) + (1 - o_0)(24 - z), (1 - o_0)(y - 24)\rangle \\ \qquad\qquad\qquad \text{if } z > 24 \text{ and } (z - 24)^2 + (y - 24)^2 \in [12, 36] \\ \langle o_0 y, o_0(-x - 2) + (1 - o_0)(-z - 24), (1 - o_0)(y - 24)\rangle \\ \qquad\qquad\qquad \text{if } z < -24 \text{ and } (z - 24)^2 + (y - 24)^2 \in [12, 36] \end{cases}$$

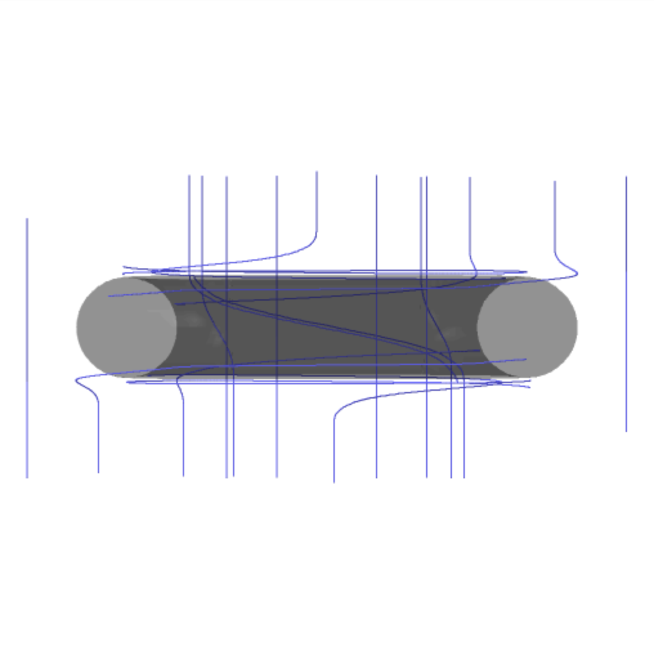

Figure 9: Trajectories near $\mathcal{O}_0$

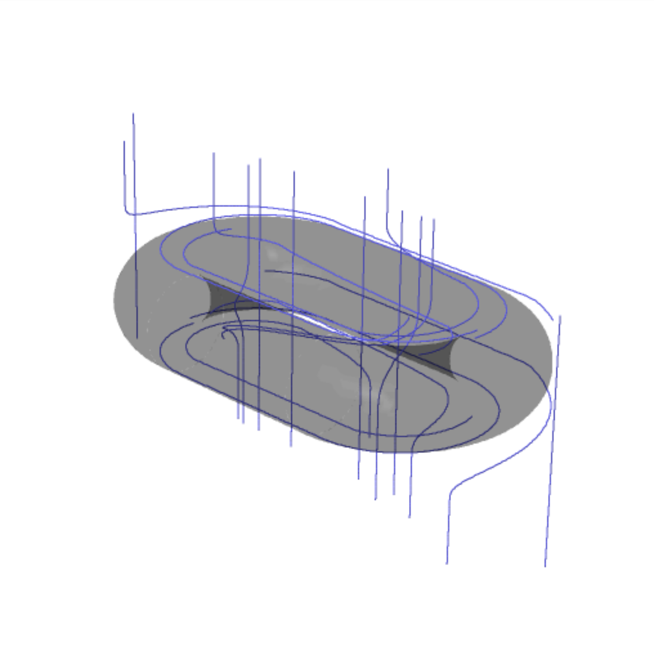

Figure 10: Trajectories near $\mathcal{O}_0$

- The bump function ensures that this flow is smooth with respect to the existing flow on $\mathcal{O}_0$.
- It is non-singular, and all trajectories are bounded within $\mathcal{O}_1$.
- The flow at distance 1 from the boundary of $\mathcal{O}_0$ is $\langle 0, 0, y - 24 \rangle$.
- As $\mathcal{O}_0$ is an attractor, this flow is not currently volume-preserving.

Diffeomorphism from torus to tobround

- Let $\mathcal{O}_n$ be a tobround with major radius M and minor radius m, and central obround O_n
- Let $\mathcal{T}_n$ be a solid torus with major radius $M(1 + \frac{2}{\pi})$ and minor radius m and central circle T_n
- Define a diffeomorphism g_n on from the central circle of $\mathcal{T}_n$ (in polar coordinates) to the central obround of $\mathcal{O}_n$ (in Cartesian coordinates).

For brevity, let $\hat{M} = \sqrt{\frac{2\pi r}{\pi+2} + M^2 - 2M}$, then

$$g_n(r,\theta) = \begin{cases} \left(-\frac{M}{\pi}(\pi+2)\theta + M, \frac{r\pi}{M(\pi+2)} + M - 1\right) \\ \qquad \text{if } \theta \in [0, \frac{2\pi}{\pi+2}) \\ \left(\hat{M}\cos(\frac{\pi+2}{\pi}\theta + \frac{\pi}{2} - 2) - M, \hat{M}\sin(\frac{\pi+2}{\pi}\theta + \frac{\pi}{2} - 2)\right) \\ \qquad \text{if } \theta \in [\frac{2\pi}{\pi+2}, \pi) \\ \left(\frac{M}{\pi}(\pi+2)(\theta-\pi) - M, \frac{r\pi}{M(\pi+2)} - M - 1\right) \\ \qquad \text{if } \theta \in [\pi, \pi(\frac{\pi+4}{\pi+2})) \\ \left(\hat{M}\cos(\frac{\pi+2}{\pi}\theta - \frac{3\pi}{2} - 4) + M, \hat{M}\sin(\frac{\pi+2}{\pi}\theta - \frac{3\pi}{2} - 4)\right) \\ \qquad \text{if } \theta \in [\pi(\frac{\pi+4}{\pi+2}), 2\pi) \end{cases}$$

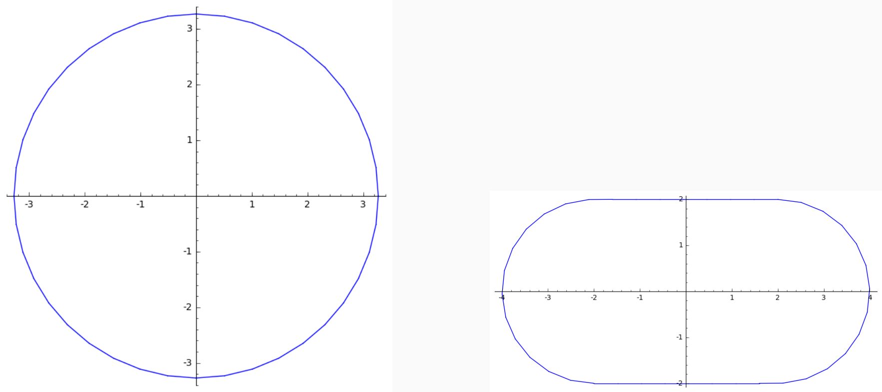

Figure 11: Central circle and central obround under g_1

- Both the torus and the tobround have the same minor radius and are each oriented in the xy-plane.

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- Extend g_n by defining $G_n : \mathcal{T}_n \to \mathcal{O}_n$ as $G_n(\mathcal{T}_n) = g_n(T_n) \times D_m$
- G_n is then a diffeomorphism. As the Jacobian of each piece of g_n has determinant 1, and the map is the identity on D_m, G_n is volume preserving.

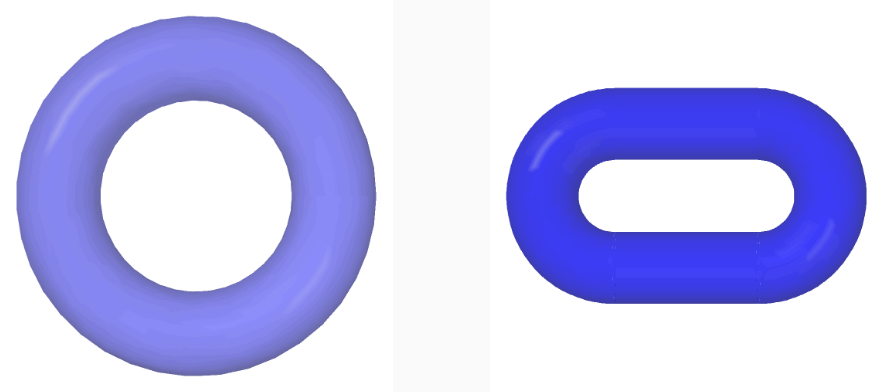

Figure 12: Torus and Tobround under G_1

Use the same flow around the torus as was constructed earlier.

$$W_{\mathcal{T}_n} = \langle 0, b_n + \frac{\partial b_n}{\partial z} r\theta(6\cdot 2^{2n-1}) - \frac{\partial b_n}{\partial z} r^2 \cos(\theta), (1 - b_n)(r\sin(\theta) - 6\cdot 2^{2n-1})\rangle.$$

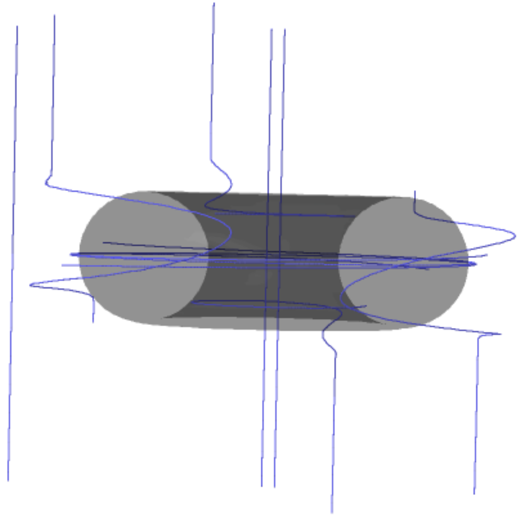

Figure 13: $W_{\mathcal{T}_0}$ around a solid torus

$$W_{\mathcal{T}_n} = \langle 0, b_n + \frac{\partial b_n}{\partial z} r\theta(6\cdot 2^{2n-1}) - \frac{\partial b_n}{\partial z} r^2 \cos(\theta), (1-b_n)(r\sin(\theta) - 6\cdot 2^{2n-1})\rangle.$$

- $W_{\mathcal{T}_n}$ is divergence free.
- $W_{\mathcal{T}_n}$ is smooth with respect to a flow on $\mathcal{T}_n$ by circular orbits.
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- Let $G_{n,*}$ be the Jacobian of G_n, then $W_n = G_{n,*}(W_{\mathcal{T}_n})$ is a divergence-free flow.

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- Let $G_{n,*}$ be the Jacobian of G_n, then $W_n = G_{n,*}(W_{\mathcal{T}_n})$ is a divergence-free flow.
- W_n is vertical at distance 1 from the boundary of $\mathcal{O}_n$.

Inserting this flow around each tobround $\mathcal{O}_n$ in our construction results in a volume-preserving flow inside of each tobround $\mathcal{O}_{n+1}$.

- This can be inserted with a rotation if n is odd
- The flow has not caused any trajectories contained in a tobround to leave that tobround, since the modification only exists up to a distance 1 from the boundary of a tobround, and the boundary of the next largest tobround is at least 2 units away.
- As the flow on this modified region agrees with the flow previously constructed at all transitions, we have a non-singular, volume-preserving dynamical system on $\mathbb{R}^3$, with all trajectories bounded.

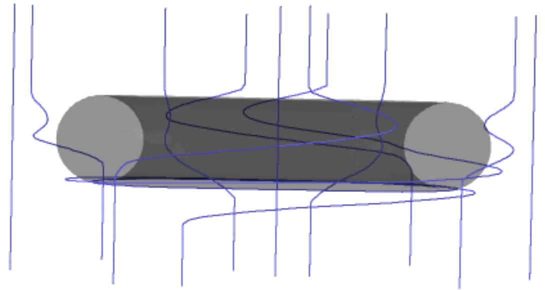

Figure 14: Trajectories near $\mathcal{O}_0$

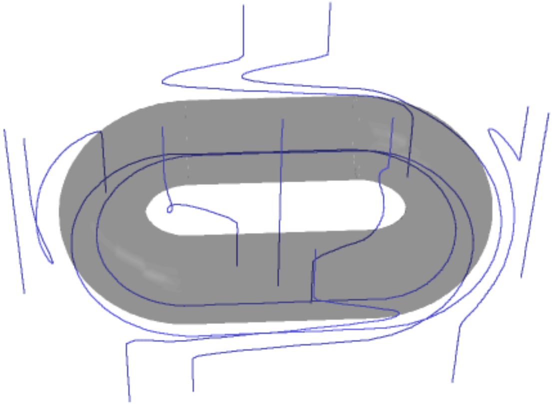

Figure 15: Trajectories near $\mathcal{O}_0$

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