IBL teaching methods in an advanced class on Vietoris Homology

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Distinguishing two topologies

- Moore Method is (historically) associated with general/point set topology.
- Adapted to areas that lend themselves to axiomatic development. (e.g. Analysis, Geometry, Set Theory).
- Axiomatic development means students develop ability and intuition by proving statements about the axioms or providing counter examples.
- Algebraic topology is strongly motivated by examples.
- Many examples used to understand theory/application of homology.

Testimonials

Some quotes: "A whole semester and I could tell that the the plane is not homeomorphic to the plane minus a point." - Unnamed set theoretic topologist.
"Yeah, but there is no theory, just examples" - Unnamed potential contributor to our text.

- A typical approach:
  Basic Euclidean topology →simplicial complexes ...
  →cycles and boundaries →homology →simplicial maps ...
  →barycentric subdivision →induced maps ...
  →distinguishing topologies of "nice" topological spaces
  ("nice" = inverse limits of simplical complexes usually related via barycentric subdivision i.e. homeomorphic).
- Potential frustrations:
  We expect spaces not to be "nice."
  (i.e. we rarely deal with finite simplicial complexes in point set topology).

Change in perspective adds additional barrier to applying intuition from point set topology.

Illustrations

(Some examples from general topology)
Graph of arc, $\sin(1/x)$ curve, pseudo-arc
Graph of circle, warsaw circle, pseudo-circle
Graph of disk, annulus, double annulus, Sierpinski curve.

Our solution

Vietoris Homology

Appeal:

- Can be developed for an arbitrary compact metric space
- Smoother transition from general topology to algebraic topology
- Ideas related to current trends in applied algebraic topology (e.g. persistence homology)

Downsides:
Relies more on a general topology background.
Computation of homology more involved.

Abstract Simplicial Complex

Definition

Let X be a non-empty set. An abstract simplicial complex of X is a collection $\mathcal{S}$ of finite nonempty subsets of X, such that

1. if A is a one element subset of X, then $A \in \mathcal{S}$,
2. if $A \in \mathcal{S}$ and B nonempty subset of A, then $B \in \mathcal{S}$.

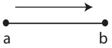
A chain α is a formal sum of finitely many simplices with coefficients in an abelian group G.

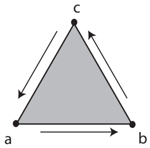
Chains, cycles, and boundaries

Let $C_p(\mathcal{S}; G)$ be all of the p-dimensional chains in $(\mathcal{S}; G)$.
We define the boundary operator $\partial : C_p(\mathcal{S}; G) \to C_{p-1}(\mathcal{S}; G)$
Given a chain, α, we denote it's boundary $\partial\alpha$.

- $Z_p(\mathcal{S}; G) = \ker(\partial)$ — cycles
- $B_p(\mathcal{S}; G) = \mathrm{im}(\partial)$ — boundaries

Problem: Compute the cycles and boundaries of the following simplicial complexes.





Vietoris Complex

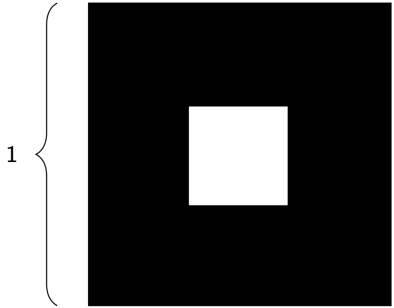
Let X be a compact, metric space, and let G be an abelian group.

Definition

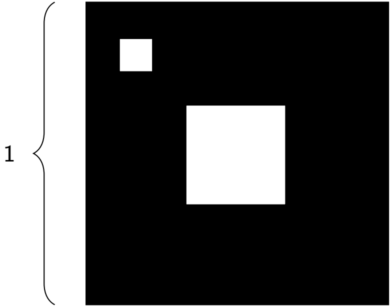
For $\varepsilon > 0$, let $\mathcal{S}(X, \varepsilon)$ be the abstract simplical complex consisting of finite subsets of diameter less than ϵ.

If $\sigma \in \mathcal{S}(X, \varepsilon)$, then σ is called an ε-simplex. The elements of $C_p(\mathcal{S}(X, \varepsilon), G)$, $Z_p(\mathcal{S}(X, \varepsilon), G)$, and $B_p(\mathcal{S}(X, \varepsilon), G)$ are called ε-chains, ε-cycles, and ε-boundaries, respectively.

Annulus Example

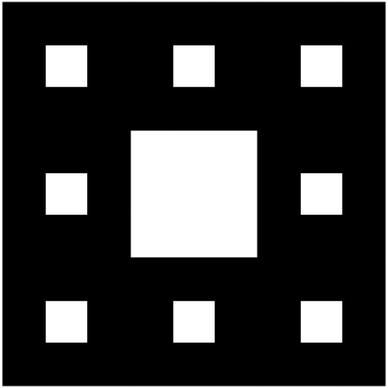


Problem: Given an annulus centered at the origin, with inner side length $1/3$ and outer side length 1, find the (greatest) number $d > 0$ such that the annulus pictured has a d-cycle that is not a boundary.



Problem: Removing a square of side length $1/9$, find the (least) number c so that for every number $c' > c$, each c-cycle is a c'-boundary.

How far can we go?



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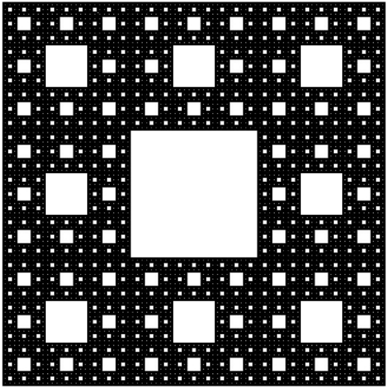
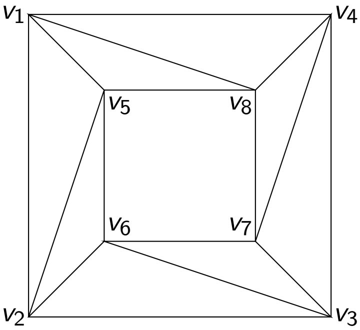


Which cycles do we count?

Definition

Two ε-cycles are homologus if their difference is a boundary.



Problem: Show that any cycle is homologous to $[v_!, v_2, v_3, v_4]$.

Definition

An infinite p-dimensional chain $\underline{\gamma}$ is a sequence of ε_n-chains γ_n such that $\lim_{n\to 0} \varepsilon_n = 0$.

Definition

An infinite p-dimensional chain $\underline{\gamma}$ is an infinite cycle if $\partial\gamma_n = 0$ for all $n = 1, 2, \ldots$.

Definition

An infinite p-dimensional chain $\underline{\gamma}$ is boundary if there exists an infinite $p+1$-dimensional chain $\underline{\omega} = \{\omega_n\}$ such that $\partial\omega_n = \gamma_n$ for all $n = 1, 2, \ldots$.

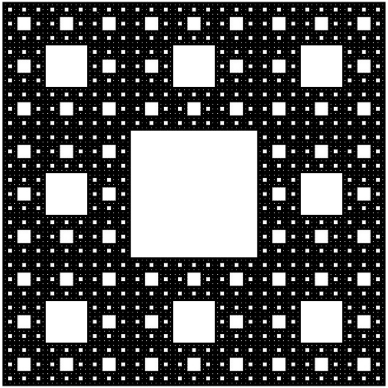
Definition

An infinite cycle $\underline{\gamma}$ is a fundamental cycle if $\gamma_n - \gamma_{n-1}$ is a boundary for all $n = 1, 2, \ldots$.

- $\check{Z}_p(X; G) = \{\underline{\alpha} : \alpha \text{ is a fundamental } p\text{-dimensional cycle}\}$
- $\check{B}_p(X; G) = \{\underline{\alpha} : \alpha \text{ is a fundamental } p\text{-dimensional cycle and a boundary}\}$
- p-th Vietoris Homology Group

$$\check{H}_p(X; G) = \frac{\check{Z}_p(X; G)}{\check{B}_p(X; G)}$$

Sierpinski Carpet



What are the Vietoris Homology groups?

1. Abstract Simplicial Complex
2. Chains, Cycles, and Boundaries
3. Homology Groups of an Abstract Simplicial Complex
4. Homology Groups of Graphs and Other Easy Examples
5. Vietoris Homology
6. Vietoris vs. Simplicial
7. Homology of Chainable Continua
8. Homology of Circularly Chainable Continua

- Appendix I: Geometric Realization of a Simplicial Complex
- Appendix II: Barycentric Subdivisions
- Appendix III: Orientation
- Appendix IV: The Pseudo-arc and the Pseudo-circle
- Appendix V: Shape of planar continua

Any Questions?