A VOLUME-PRESERVING DYNAMICAL SYSTEM ON $\mathbb{R}^3$ WITH ALL TRAJECTORIES BOUNDED

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Abstract. We present here a smooth construction of a non-singular, volume-preserving dynamical systems on $\mathbb{R}^3$, with each trajectory contained in a bounded set. This is achieved by nesting a sequence of subsets of $\mathbb{R}^3$, so that any trajectory originating in a particular subset stays in that subset. The vector fields that achieve this are then modified to preserve volume.

Theorem 1. *There exists a smooth, non-singular, volume-preserving dynamical system on $\mathbb{R}^3$, with all trajectories bounded.*

This theorem arises from a question asked by G. Kuperberg, following his work in [4]. As reported in [3], Kuperberg asked if there existed a non-singular dynamical system on $\mathbb{R}^3$, which was volume-preserving, and in which all trajectories were uniformly bounded.

It is shown in [2] that there exists a non-singular flow on $\mathbb{R}^3$ with all trajectories bounded, though the bound is not uniform. As a solution to Problem 110 in [5], it was shown by Kuperberg and Reed, that there did exist a non-singular dynamical system on $\mathbb{R}^3$ with each trajectory uniformly bounded.

While G. Kuperberg's original question remains open, we present here progress towards a solution in the form of Theorem 1. We begin with the notion of nested tori used in [2]. Jones and Yorke used nested tori to achieve the bounded trajectories portion of our results. This can be made volume preserving by changing the flow and deforming the tori via a diffeomorphism.

We recall a few facts about dynamical systems. A *dynamical system* is a triple $(\mathbb{R}, \Omega, \pi)$, with Ω a topological space and $\pi : \mathbb{R} \times \Omega \to \Omega$, such that π is continuous. We need the additional properties that, for any $x \in \Omega$, and any $t_1, t_2 \in \mathbb{R}$, we have $\pi(0, x) = x$ and $\pi(t_1, \pi(t_2, x)) = \pi(t_1 + t_2, x)$.

If π is a C^r function, we say the dynamical system is also C^r. For simplicity, we denote $\pi(t, x)$ as tx.

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $(\Omega, \mathbb{R}, \varphi)$ be a dynamical system as defined above. We define the inverse of φ by letting $\varphi^{-1}(t, x) = \varphi(-t, x)$. Let φ have the added condition that, for all $A \in \mathcal{A}$, $\varphi^{-1}(A) \in \mathcal{A}$. The measure μ is *invariant* with respect to φ if $\mu(A) = \mu(\varphi^{-1}(A))$ for all $A \in \mathcal{A}$. If μ is invariant with respect to φ, $(\Omega, \mathcal{A}, \mu, \varphi)$ is a *measure-preserving dynamical system.*

Any Riemannian volume form gives rise to a measure which is a non-zero scalar multiple of Lebesgue measure [**?**]. n the case that a measure μ given by a smooth Riemannian volume form and the dynamical system is parallel to a smooth vector field $\vec{v}$ on $\mathbb{R}^n$, the measure-preserving condition is equivalent to the divergence equation [**?**]

$$\nabla \cdot \vec{v} = 0.$$

Such a dynamical system is called *volume-preserving.*

Take a manifold M, a measure μ on M, and a dynamical system $(\mathbb{R}, M, \pi)$, which is not necessarily measure preserving. The *likely limit set* $\mathcal{A}(M)$ of this dynamical system is the smallest closed subset of M such that $\omega(x) \subset \mathcal{A}(M)$, for all x in M, excluding a set of measure zero [6].

As stated by Milnor [6], if $(\mathbb{R}, M, \pi)$ is measure-preserving, then $\mathcal{A}(M) = M$. By contrapositive, if $\mathcal{A}(M) \neq M$, then $(\mathbb{R}, M, \pi)$ is not measure-preserving. This motivates our constructions. We seek dynamical systems where all likely limit sets have a pre-image with measure zero.

We start with a theorem of Jones and Yorke [2] to motivate our constructions.

Theorem 2. *There exists a non-singular flow on $\mathbb{R}^3$, with all trajectories bounded* [2].

Proof. Define the function $c(r) = \frac{2}{3}(4^{r+1} - 4)$. Construct a set of tori, $\{T^r : r \in 0, 1, 2, \dots\}$, where T^r is the region bounded by the parametric surface

$$\begin{aligned}
&x = 4^r(2 + \cos(u))\cos(v)\\
&y = 4^r(2 + \cos(u))\sin(v) + c(r)\\
&z = 4^r \sin(u)\\
&\text{ if } r \text{ is even, and}\\
&x = 4^r \sin(u)\\
&y = 4^r(2 + \cos(u))\sin(v) + c(r)\\
&z = 4^r(2 + \cos(u))\cos(v)\\
&\text{ if } r \text{ is odd}
\end{aligned}$$

for $u, v \in [0, 2\pi]$.

Each solid torus T^i is then completely contained in T^{i+1}, with the even indexed tori in the xy-plane, and the odd indexed tori in the yz-plane. Each T^r is centered at $(0, c(r), 0)$. The nesting of the first three tori is shown in Figure 1.

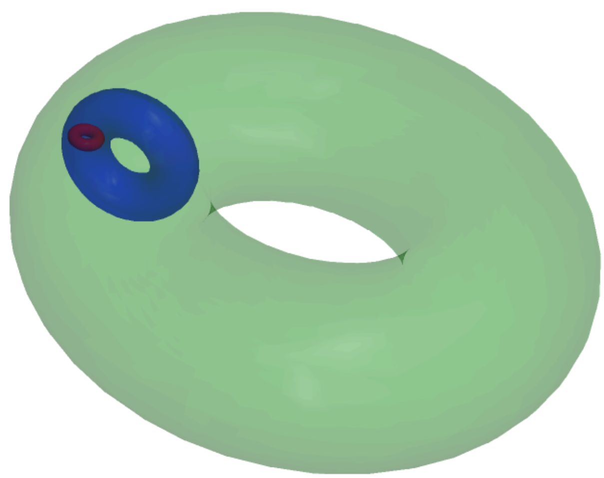


FIGURE 1. Nested tori in $\mathbb{R}^3$

Let $p = (x, y, z)$ be a point in $\mathbb{R}^3$. Let $i(r) = \frac{1-(-1)^r}{2}$ so that the function i returns 0 if r is even, and 1 if r is odd. Let $G_0(p) = (y, -x, 0)$ and $G_1(p) = (0, -z, y)$. Let $h_0(p) =$max$\{0,$min$\{1, 1 - d(T^0, p)\}\}$ and $h_r(p) =$max$\{0,$min$\{1, 1 - d(T^r, p), d(T^{r-1}, p)\}\}$.

Define a flow on $\mathbb{R}^3$, by

$$\dot{p} = \sum_{r=0}^{\infty} G_{i(r)}(p - (0, c(r), 0))h_r(p). \tag{1}$$

Clearly all trajectories in T^0 are circles, and so remain bounded in T^0. We can also visualize the flow near the boundary of T^0 with trajectories through T^1, as shown in Figure 2.

For any point in T^n, the omega limit set of that point is in T^n, either by remaining in a periodic orbit in T^n, or by attracting to the surface of T^{n-1}. Hence, each T^n is invariant, and as $\mathbb{R}^3 = \cup_{n=0}^{\infty} T^n$, all trajectories are bounded.

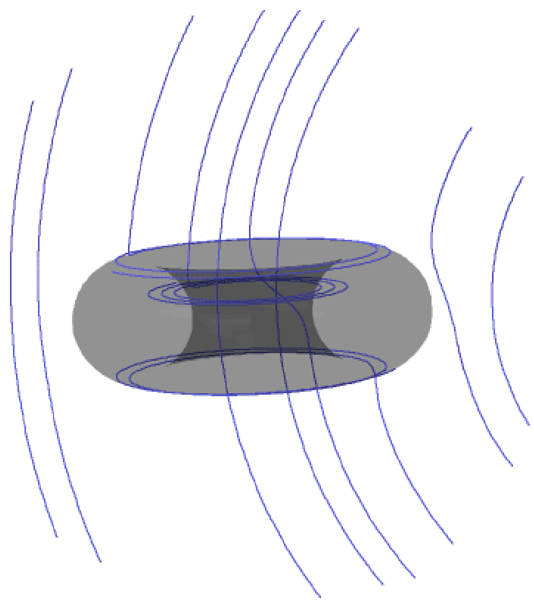


FIGURE 2. Flow near the boundary of T^0

We now show that the flow is non-singular. We first show that all solutions of (1) are unique.

For each natural r, if $p \in T^r$, $h_m(p) = 0$ for all $m > r$. Therefore, (1) is always computed as a finite sum. (1) is therefore locally Lipschitz on each T^r, and as $\mathbb{R}^3 = \cup_{i\in\mathbb{N}} T^r$, it is locally Lipschitz on $\mathbb{R}^3$. Therefore, solutions of (1) exist are are unique [1].

To verify that $\dot{p} \neq 0$, for all $p \in \mathbb{R}^3$, first fix p and let $r = \min\{s : x \in T^s\}$. As above, $h_m(p) = 0$ for all $m > r$. If $d(p, T^{r-1})n \geq 1$, then $h_r(p) = 1$. If $0 < d(p, T^{r-1}) < 1$, then $h_r(p) = h_{r-1}(p) = d(p, T^{r-1}) \neq 0$. If $m < r - 1$, $d(p, T^m) > d(p, T^r)$, which implies $1 - d(p, T^m) < 0$, and therefore $h_m(p) = 0$. We need only consider the r and $r - 1$ terms of the sum to determine that the flow is non-singular.

If $r = 0$, $\dot{p} = (y, -x, 0)$, which is non-zero for all points in T^0. If $r > 0$, $\dot{p} = (y - c(r), -x - z, y - c(r-1)$ when r is even, and $\dot{p} = (y - c(r-1), -x - z, y - c(r)$ when r is odd. In either case, $\dot{p} = 0$ iff $y = c(r-1)$ and $y = c(r)$. By the definition of $c(r)$, this is never true, and we conclude that the flow is non-singular. □

As constructed, this flow is not measure-preserving. The boundary of T^n is an attractor for a subset of T^{n+1} of non-zero measure, and by a result of Milnor [6], cannot preserve measure. This can also be verified directly by writing the function h explicitly for points whose distance from T^0 is less than or equal to 1. It is easy to check that the explicit construction of this flow has non-zero divergence.

As a warmup to our main result, we prove a C^0 version. This modifies the Jones and Yorke [**?**] construction in theorem 2 to cause it to preserve volume. In doing so we will lose smoothness, but we can regain it later in the proof of theorem 1.

Theorem 3. *There exists a C^0, non-singular, volume-preserving dynamical system on $\mathbb{R}^3$, with all trajectories bounded.*

Proof. Let $\vec{F}$ be the flow in Theorem 2. We will construct a volume-preserving semi-plug containing a torus, with the flow on the torus composed of entirely circular orbits.

We can then modify the flow in a neighborhood of each T_n, which is contained in T_{n+1}, and which does not affect the flow on the boundary of T_{n+1} or T_n.

Define a function

$$h_n(r,z) = z^2 + (r - 2 \cdot 4^n)^2 - 4^n.$$

Note that this is the square of the distance function from a point to the boundary of T_n.

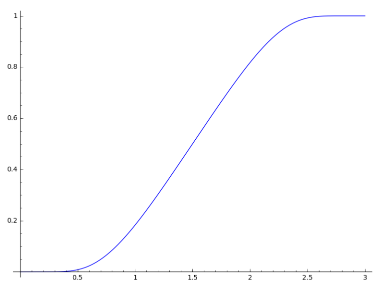
We will need a smooth ramp functions in our construction, as well as its derivative. Begin with

$$b(x) = e^{-1/x}.$$

We then define

$$b_1(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{b(x/3)}{b(x/3)+b(1-x/3)} & \text{if } x \in (0,3) \\ 1 & \text{if } x \ge 3 \end{cases}$$

$$b_2(x) = \frac{db_1}{dx} = \begin{cases} 0 & \text{if } x \le 0 \text{ or } x \ge 3 \\ \frac{b(x/3)b(1-x/3)[\frac{9}{x^2}+\frac{9}{(x-3)^2}]}{3[b(x/3)+b(1-x/3)]^2} & \text{if } x \in (0,3) \end{cases}$$



(A) Ramp function b_1

(B) Bump function b_2, the derivative of b_1

FIGURE 3. The function b_1 and its derivative b_2.

Let $\mathfrak{C}_n$ be a cylinder with radius $3 \cdot 4^n + 1$ and height $2 \cdot 4^n + 2$. Note that a torus with the same dimensions as T_n, centered at the origin and lying in the xy-plane, will fit inside $\mathfrak{C}_n$.

Construct a vector field P_n in cylindrical coordinates (r, θ, z) on $\mathfrak{C}_n$ such that

(1) P_n is non-singular.
(2) As $h_n \to 0$, $P_n \to \langle 0, -1, 0 \rangle$
(3) As $h_n \to 3$, $P_n \to \langle 0, 0, -1 \rangle$
(4) P_n is divergence-free.

Define P_n in cylindrical coordinates (r, θ, z). Let $h = h_n(r,z)$, and note that $h = 0$ for points on the boundary of T_n and returns 3 for points distance 1 from the boundary of T_n. Then

$$P_n = \begin{cases} \langle 0, -1, 0 \rangle & \text{if } h \le 0 \\ \langle zrb_2(h), -1 + b_1(h), -b_1(h) - r(r-2)b_2(h) \rangle & \text{if } h \in (0,3) \\ \langle 0, 0, -1 \rangle & \text{if } h \ge 3 \end{cases}$$

Each function in P_n is always defined when $h_n \in [0,3]$, and $P_n \neq \langle 0,0,0 \rangle$ for all points in $\mathfrak{C}_n$. As all partial derivatives of the components of P_n are bounded (by construction of the bump functions), the differential equation $\dot{p} = P_n$ has a unique solution [1]. Therefore, P_n is non-singular, and condition 1 is satisfied.

Looking at the behavior of P_n as h changes, we satisfy conditions 2 and 3, since

- As $h \to 0$, $b_1(h) \to 0$, $b_2(h) \to 0$
- As $h \to 3$, $b_1(h) \to 1$, $b_2(h) \to 0$

In cylindrical coordinates, denote the components of the vector field $P_n = \langle P_r, P_\theta, P_z\rangle$. The divergence equation in cylindrical coordinates yields

$$\begin{aligned}
\nabla \cdot P_n &= \frac{1}{r}\left(\frac{\partial}{\partial r} r P_r\right) + \frac{1}{r}\left(\frac{\partial}{\partial \theta} P_\theta\right) + \frac{\partial}{\partial z} P_z \\
&= \frac{1}{r}\left(\frac{\partial}{\partial r} z r^2 b_2(h)\right) + \frac{1}{r}\frac{\partial}{\partial \theta}(-1 + b_1(h)) + \frac{\partial}{\partial z}(-b_1(h) - r(r-2)b_2(h)) \\
&= \frac{1}{r}\left(2zrb_2(h) + zr^2\frac{\partial b_2}{\partial h}\frac{\partial h}{\partial r}\right) + 0 - \frac{\partial b_1}{\partial h}\frac{\partial h}{\partial z} - r(r-2)\frac{\partial b_2}{\partial h}\frac{\partial h}{\partial z} \\
&= 2zb_2(h) + zr\frac{\partial b_2}{\partial h}(2(r-2)) - b_2(h)(2z) - r(r-2)\frac{\partial b_2}{\partial h}(2z) \\
&= 0
\end{aligned}$$

and we have satisfied all 4 conditions. We have two components to the boundary of the cylinder. The vector field is always $\langle 0, 0, -1\rangle$. This flow is on a cylinder, so we can think of the boundary at the top and bottom as disks, on which the flow is either transverse in, or transverse out. On the sides of the cylinder, the flow is just vertical trajectories.

For each n, let $\hat{T_n} = \{(r, \theta, z) \in \mathbb{R}^3 : h_n(r, z) \in (0, 1)\}$. Since $\mathfrak{C}_n$ contains a copy of T_n, it also contains a copy of $\hat{T_n}$. We can therefore consider $\hat{P_n}$ to be the portion of P_n restricted to $\hat{T_n}$.

We insert each $\hat{P_n}$ into $\vec{F}$ in two different ways. If n is even, insert $\hat{P_n}$ into $\vec{F}$ by translating $c(n)$ units along the $y-$axis. If n is odd, inset $\hat{P_n}$ into $\vec{F}$ by translating $c(n)$ units along the $y-$axis and rotating $\hat{P_n}$ about the $x-$axis by an angle of $\pi/2$. In either case, the Dehn surgery in the insertion is performed with slope 1, so the topology of the underlying manifold (in this case, $\mathbb{R}^3$, is not changed.

Denote the modified vector field on $\mathbb{R}^3$ by $\vec{W}$.

FIGURE 4. $\vec{W}$ in a neighborhood of T_0

We now verify that the solution to differential equation $\dot{p} = \vec{W}$ satisfies the conditions of Theorem 3.

It follows from the facts that $\vec{F}$ and P_n are each non-singular that $\vec{W}$ is non-singular.

The partial derivatives of P_n are all bounded, and therefore P_n satisfies the Lipschitz condition, and the equation $\dot{p} = P_n$ has a solution.

Let S_n be the set of all points p such that $h_n(p) = 1$. Denote this the *switching manifold* on T_n [**?**]. This is the set of points where the insertion of $\hat{P_n}$ meets the existing vector field $\vec{F}$. In our definition of insertion above, we considered smoothly changing the lengths of the vectors in a neighborhood of this switching manifold. Here instead, we can consider a piecewise-smooth construction.

We can define the *degree of smoothness* [**?**] at each point $p \in S_n$, as the highest order r such that the Taylor Series expansions of the dynamical systems determined by the vector fields on either side of the switching manifold, evaluated at zero, agree up to the $(r-1)$-st term. Our vector fields are not gradient fields, but they do agree after one partial derivative is taken. Therefore, the degree of smoothness in our switching manifold is zero, and $\vec{W}$ is C^0.

As translations and rotations are volume preserving, and those are the only operations needed during our insertions, the vector field $\vec{W}$ is divergence-free whenever P_n is divergence free.

That fact that all trajectories are bounded follows immediately the embedding of P_n, which will touch the boundary of T_n only in the case of T_1, but in this case, the embedded vector field agrees with $\vec{F}$.

In all other cases, the embedded field does not touch the boundary of T_{n+1}, hence the property of bounded trajectories from [2] is preserved.

We conclude that the solution to the differential equation $\dot{p} = \vec{W}$ is a non-singular, measure-preserving dynamical system on $\mathbb{R}^3$, with all trajectories bounded. $\square$

We need one additional definition and an additional lemma before we proceed.

Given a diffeomorphism between two manifold $g : M \to N$, and a vector field $\vec{V}$ on M, let $g_*(\vec{V})$ be the vector field on N given by operating on each vector in $\vec{V}$ by the Jacobian of g.

Lemma 4. *If $\vec{V}$ is a vector field that yields a volume-preserving dynamical system on a manifold M, and $g : M \to N$ is a diffeomorphism whose Jacobian has determinant of absolute value 1 at all points in M, then $g_*(\vec{V})$ yields a volume-preserving dynamical system on N* [**?**].

We can now prove Theorem 1.

Proof. Define an *obround* of radius R as the set of points in $\mathbb{R}^2$

$$\{(x, y) \in \mathbb{R}^2 : x \in [-R, R], y = \pm R\} \cup \{(x, y) \in \mathbb{R}^2 : (x \pm 2R)^2 + y^2 = R^2\}.$$

An obround is the boundary of a square of side length $2r$, with semi circles of radius r appended to the right and left sides. This is shown in Figure 5.

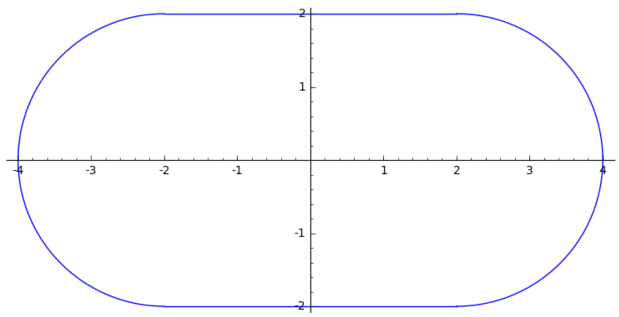


FIGURE 5. Obround with radius 2

A *tobround* of major radius R and minor radius r, with $r < R$, is the Cartesian product of a solid disk of radius r and an obround of radius R. This is shown in Figure 6.

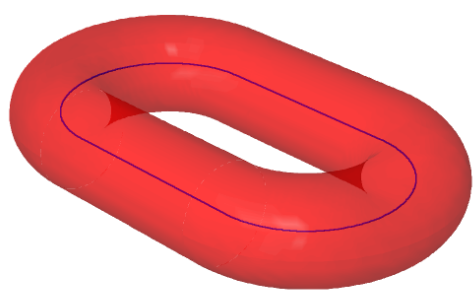


FIGURE 6. Tobround with major radius 2, and minor radius 1

The process of constructing the flow that satisfies this theorem is as follows:

(1) Construct a nested sequence of tobrounds, whose union is $\mathbb{R}^3$.
(2) Define a flow in each tobround with non-singular trajectories that are contained in the tobround in which they originate.
(3) The distance from the boundary of a tobround to the boundary of the larger tobround in which it is nested must always be greater than or equal to 2.
(4) Construct a volume-preserving diffeomorphism from a solid torus to a tobround for each tobround in the construction.

(5) Construct a smooth, volume-preserving flow around the torus, which is vertical at distance 1 from the boundary of the torus.
(6) Apply the Jacobian of the diffeomorphism from point 4 above to the vector field around the torus to obtain a flow in a neighborhood of each tobround.
(7) The resulting flow agrees with the existing flow on the nested tobrounds, is non-singular, has all trajectories trapped within a particular tobround, and preserves volume.

To define a sequence of tobrounds whose union is $\mathbb{R}^3$, let $\mathcal{O}_0$ be a tobround with minor radius 1 and major radius 2. If $n > 0$ is even, let $\mathcal{O}_n$ be the tobround with minor radius $6 \cdot 2^{2n-1}$ and major radius $6 \cdot 2^{2n}$, shifted $6 \cdot 2^{2n}$ units positively along the y-axis. If $n > 0$ is odd, the major and minor radii are as defined above, as is the shifting along the y-axis, but the entire obround is rotated about the $y-$axis by an angle of $\pi/2$. This is shown in Figure 7.

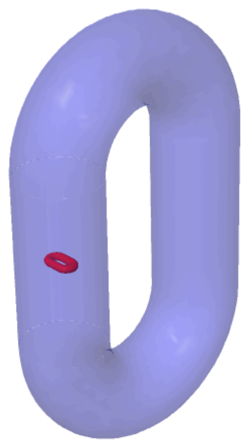


FIGURE 7. $\mathcal{O}_0$ nested in $\mathcal{O}_1$

Note that

$$\bigcup_{n\in\mathbb{N}} \mathcal{O}_n = \mathbb{R}^3.$$

We can construct a flow on $\mathbb{R}^3$ by defining it on each tobround. For each $p \in \mathbb{R}^3$, let $d(p, \mathcal{O}_n)$ be the usual distance function. For each $\mathcal{O}_n$, let $o_n(d(p, \mathcal{O}_n)$ be the smooth bump function which returns 1 on the boundary of $\mathcal{O}_n$ and 0 for all points whose distance from $\mathcal{O}_n$ is greater than or equal to 1.

Define a piecewise vector field in Cartesian coordinates on $\mathcal{O}_0$ by

$$\dot{p} = \begin{cases} \langle y, 0, 0\rangle & \text{if } x \in [-2, 2] \\ \langle y, 2-x, 0\rangle & \text{if } x > 2 \text{ and } (x-2)^2 + y^2 \in [1, 9] \\ \langle y, -x-2, 0\rangle & \text{if } x < -2 \text{ and } (x+2)^2 + y^2 \in [1, 9] \end{cases}$$

The trajectories are then clockwise oriented obrounds, within the tobround $\mathcal{O}_\prime$. Sample trajectories are shown in Figure 8. This is clearly non-singular, and all trajectories are bounded.

Let o_0 be the bump function above, with argument assumed to be $d(p, \mathcal{O}_0)$. Extend our flow to $\mathcal{O}_1$ by

$$\dot{p} = \begin{cases} \langle o_0 y, 0, (1-o_0)(y-24)\rangle \\ \qquad \text{if } z \in [-24, 24] \\ \langle o_0 y, o_0(2-x) + (1-o_0)(24-z), (1-o_0)(y-24)\rangle \\ \qquad \text{if } z > 24 \text{ and } (z-24)^2 + (y-24)^2 \in [12, 36] \\ \langle o_0 y, o_0(-x-2) + (1-o_0)(-z-24), (1-o_0)(y-24)\rangle \\ \qquad \text{if } z < -24 \text{ and } (z-24)^2 + (y-24)^2 \in [12, 36] \end{cases}$$

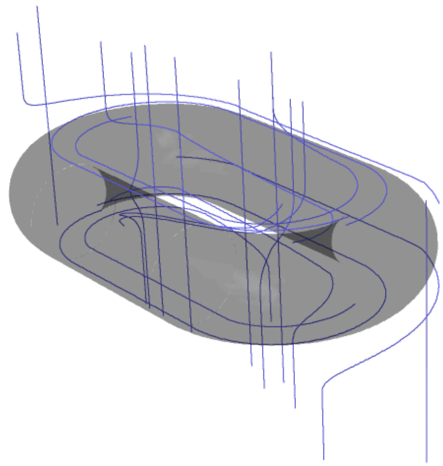


FIGURE 8. Trajectories near $\mathcal{O}_0$

This gives us the first steps towards an analogous result to Jones and Yorke, but with nested tobrounds. The bump function ensures that this flow is smooth with respect to the existing flow on $\mathcal{O}_0$. It is non-singular, and all trajectories are bounded within $\mathcal{O}_1$. The flow at distance 1 from the boundary of $\mathcal{O}_0$ is $\langle 0, 0, y-24\rangle$. As $\mathcal{O}_0$ is an attractor, this flow is not currently measure-preserving.

To generalize this, assume n is odd, as the case that n is even immediately follows. Then the vector field on $\mathcal{O}_n$ is given by

$$\dot{p} = \begin{cases} \langle o_{n-1}(y-6\cdot 2^{2n-1}), 0, (1-o_{n-1})(y-6\cdot 2^{2n})\rangle \\ \qquad\qquad\qquad \text{if } z \in [-6\cdot 2^{2n}, 6\cdot 2^{2n}] \\ \langle o_{n-1}(y-6\cdot 2^{2n-1}), \\ \quad o_{n-1}(6\cdot 2^{2n-1}-x) + (1-o_{n-1})(6\cdot 2^{2n}-z), \\ \qquad (1-o_{n-1})(y-6\cdot 2^{2n})\rangle \\ \text{if } z > 6\cdot 2^{2n} \text{ and } (z-6\cdot 2^{2n})^2 + (y-6\cdot 2^{2n})^2 \in [6\cdot 2^{2n-1}, 18\cdot 2^{2n-1}] \\ \langle o_{n-1}(y-6\cdot 2^{2n-1}), \\ \quad o_{n-1}(-x-6\cdot 2^{2n-1}) + (1-o_{n-1})(-z-6\cdot 2^{2n}), \\ \qquad (1-o_{n-1})(y-6\cdot 2^{2n})\rangle \\ \text{if } z < -6\cdot 2^{2n} \text{ and } (z-6\cdot 2^{2n})^2 + (y-6\cdot 2^{2n})^2 \in [6\cdot 2^{2n-1}, 18\cdot 2^{2n-1}] \end{cases}$$

In order to make this volume-preserving, we insert a vector field around each tobround, which agrees with the existing flow on the boundary of the tobround, and with the vertical flow at distance 1 from the boundary of each tobround.

This vector field is built by first constructing a volume-preserving flow around a solid torus, then use a volume-preserving diffeomorphism between the torus and the tobround to get the appropriate flow around the tobround.

For a given $\mathcal{O}_n$, denote the major radius M and minor radius m, and central obround O_n Let $\mathcal{T}_n$ be a solid torus with major radius $M(1+\frac{2}{\pi})$ and minor radius m and central circle T_n

Define a diffeomorphism g_n on from the central circle of $\mathcal{T}_n$ (in polar coordinates) to the central obround of $\mathcal{O}_n$ (in Cartesian coordinates).

For brevity, let $\hat{M} = \sqrt{2r + M^2 - \frac{2}{\pi}(\pi+2)M}$, then

$$g_n(r,\theta) = \begin{cases} (-\frac{M}{\pi}(\pi+2)\theta + M, \frac{r\pi}{M(\pi+2)} + M - 1) \\ \qquad \text{if } \theta \in [0, \frac{2\pi}{\pi+2}) \\ (\hat{M}\cos(\frac{\pi+2}{\pi}\theta + \frac{\pi}{2} - 2) - M, \hat{M}\sin(\frac{\pi+2}{\pi}\theta + \frac{\pi}{2} - 2)) \\ \qquad \text{if } \theta \in [\frac{2\pi}{\pi+2}, \pi) \\ (\frac{M}{\pi}(\pi+2)(\theta-\pi) - M, \frac{r\pi}{M(\pi+2)} - M - 1) \\ \qquad \text{if } \theta \in [\pi, \pi(\frac{\pi+4}{\pi+2})) \\ (\hat{M}\cos(\frac{\pi+2}{\pi}\theta - \frac{3\pi}{2} - 4) + M, \hat{M}\sin(\frac{\pi+2}{\pi}\theta - \frac{3\pi}{2} - 4) \\ \qquad \text{if } \theta \in [\pi(\frac{\pi+4}{\pi+2}), 2\pi) \end{cases}$$

Both the torus and the tobround have the same minor radius and are each oriented in the xy-plane. Let D_m be a solid 2-disk of radius m, then

- $\mathcal{T}_n = T_n \times D_m$.
- $\mathcal{O}_n = O_n \times D_m$.

Extend g_n by defining $G_n : \mathcal{T}_n \to \mathcal{O}_n$ as $G_n(\mathcal{T}_n) = g_n(T_n) \times D_m$. G_n is then a diffeomorphism. As the Jacobian of each piece of g_n has determinant 1, and the map is the identity on D_m, G_n is volume preserving.

Now we define a flow in a neighborhood of $\mathcal{T}_n$ using cylindrical coordinates. This is the nearly the same flow used in the proof of Theorem 3. The main difference is that we replace our distance function h with

$$\tilde{h}(r,z) = (r - 6 \cdot 2^{2n})^2 + z^2 - 6 \cdot 2^{2n-1}.$$

The bump functions $b_1(\tilde{h})$ and $b_2(\tilde{h})$ are the same as before.

The flow is then given by

$$P_{\mathcal{T}_n} = \langle 6 \cdot 2^{2n-1} zrb_2(\tilde{h}), -1 + b_1(\tilde{h}) - 2b_2(\tilde{h})zr\cos(\theta), -b_1(\tilde{h})(r\sin(\theta) - 6 \cdot 2^{2n-1}) - r(-6 \cdot 2^{2n})b_2(\tilde{h})\rangle.$$

We have the following necessary conditions which are satisfied:

(1) $P_{\mathcal{T}_n}$ is divergence free.
(2) $P_{\mathcal{T}_n}$ is smooth with respect to a flow on $\mathcal{T}_n$ by circular orbits.
(3) $P_{\mathcal{T}_n}$ is vertical at distance 1 from the boundary of $\mathcal{T}_n$.
(4) $P_n = G_{n,*}(P_{\mathcal{T}_n})$ is a divergence-free flow.
(5) The extended diffeomorphism G_n is the identity on the z-coordinate, P_n is vertical at distance 1 from the boundary of $\mathcal{O}_n$.

Each of these is quickly checked. We can verify 1, using the same notation for the components as in the proof of 2.4, via

$$\begin{aligned} \nabla \cdot P_n &= \frac{1}{r}\left(\frac{\partial}{\partial r} rP_r\right) + \frac{1}{r}\left(\frac{\partial}{\partial \theta} P_\theta\right) + \frac{\partial}{\partial z} P_z \\ &= \frac{1}{r}\left(\frac{\partial}{\partial r} 6 \cdot 2^{2n-1} zr^2 b_2(\tilde{h})\right) + \frac{1}{r}\left(\frac{\partial}{\partial \theta}\left(-1 + b_1(\tilde{h}) - 2b_2(\tilde{h})zr\cos(\theta)\right)\right) \\ &\quad + \frac{\partial}{\partial z}\left(-b_1(\tilde{h})(r\sin(\theta) - 6 \cdot 2^{2n-1}) - r(-6 \cdot 2^{2n})b_2(\tilde{h})\right) \\ &= \frac{1}{r}\left(6 \cdot 2^{2n} zrb_2(\tilde{h}) + 2r(r - 6 \cdot 2^{2n})z\frac{\partial b_2}{\partial h}\right) + 2zrb_2(\tilde{h})\sin(\theta) - 2zrb_2(\tilde{h})\sin(\theta) \\ &\quad + 6 \cdot 2^{2n} zb_2(\tilde{h}) - 2r(r - 6 \cdot 2^{2n-1})\frac{\partial b_2}{\partial h} z = 0 \end{aligned}$$

For 2 and 3, as $\tilde{h} \to 0$, the flow approaches $\langle 0, -1, 0\rangle$, and as $\tilde{h} \to 1$, the flow approaches $\langle 0, 0, -1\rangle$. Conditions 4 and 5 follow from the construction of g_n and G_n respectively.

Inserting this flow around each tobround $\mathcal{O}_n$ in our construction results in a volume-preserving flow inside of each tobround $\mathcal{O}_{n+1}$. This can be inserted with a rotation if n is odd. The flow has not caused any trajectories contained in a tobround to leave that tobround, since the modification only exists up to a distance 1 from the boundary of a tobround, and the boundary of the next largest tobround is at least 2 units away. As the flow on this modified region agrees with the flow previously constructed, and the modification is C^∞, we have a C^∞, non-singular, volume-preserving dynamical system on $\mathbb{R}^3$, with all trajectories bounded. $\square$

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