Tilings and Tiling Spaces

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Gustavus Adolphus College

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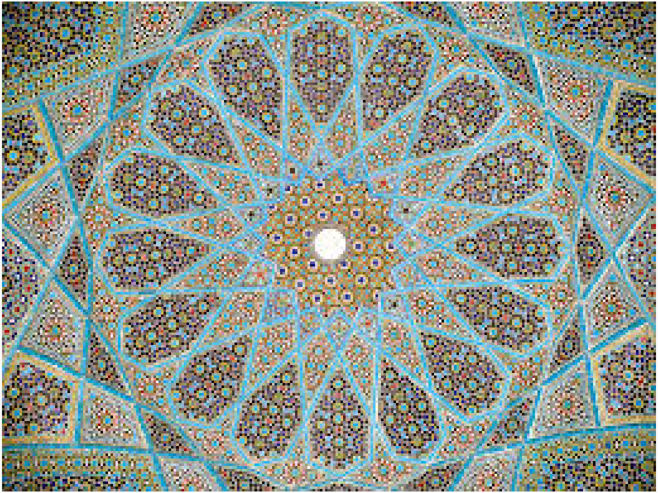
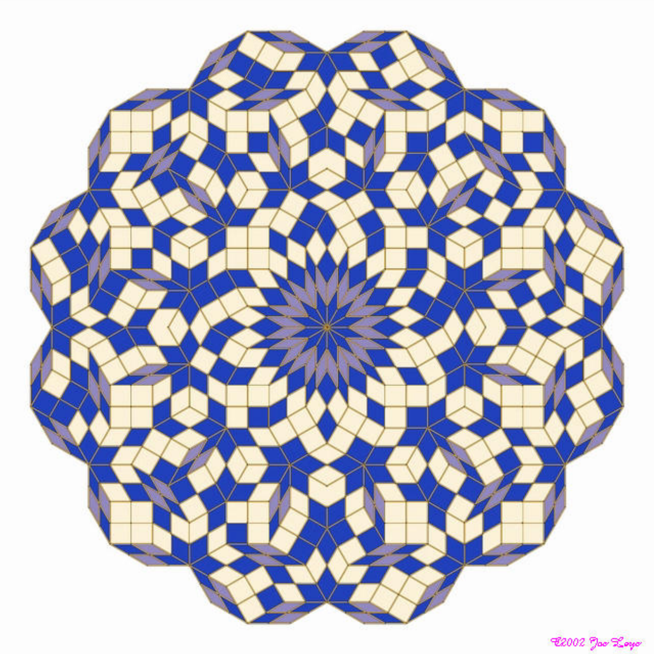
What is a tiling?

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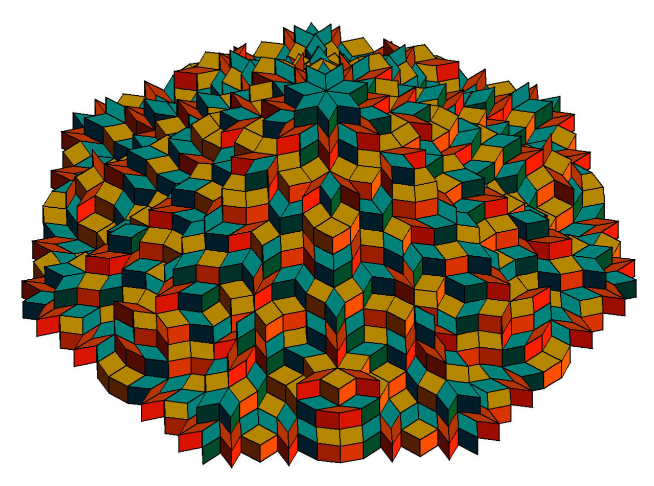

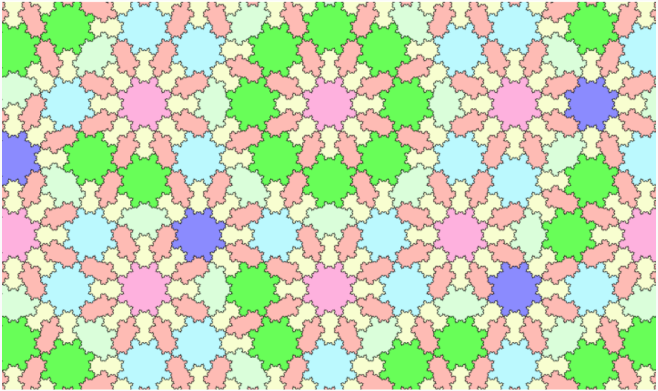

Tilings

Definition

A **tiling** T of $\mathbb{R}^n$ of a countable set $\{t_1, t_2, \ldots\}$ of subsets of $\mathbb{R}^n$ called **tiles**, such that

- Each tile is homoemorphic to a closed ball
- All tiles are pairwise disjoint
- The union of all tiles is $\mathbb{R}^n$

One-dimensional tilings

Use 2 tiles, let a be an interval of length $\frac{1+\sqrt{5}}{2}$ and let b be an interval of length 1.

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Replace a by ab and b by a.

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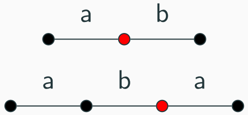

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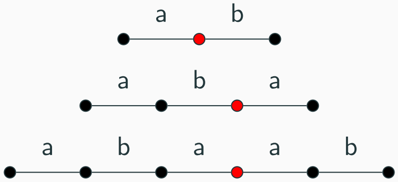

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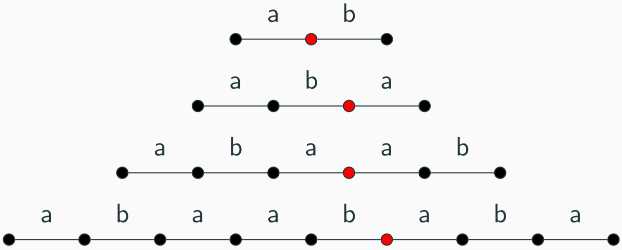
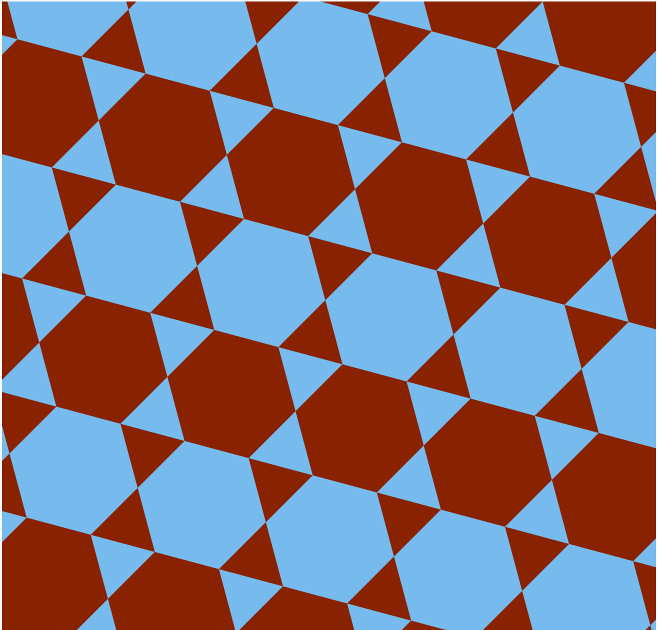
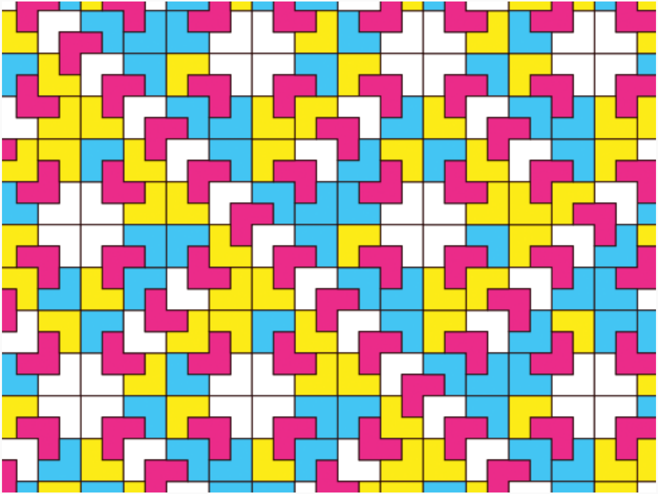


Periodic tilings of the plane



Aperiodic tilings of the plane



What makes a tiling aperiodic?

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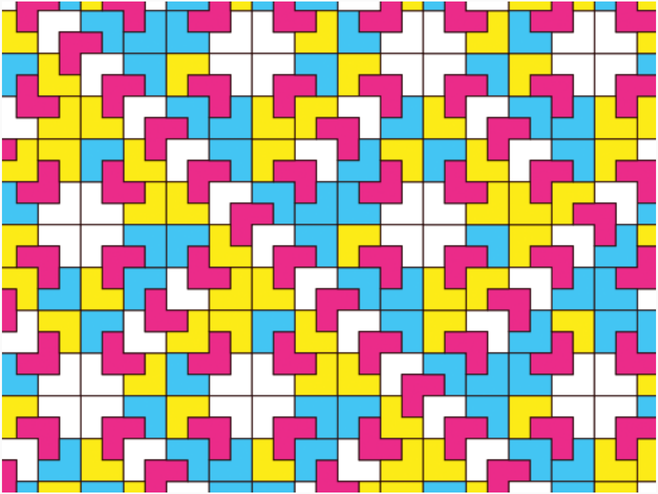
- If T is a tiling and $x \in \mathbb{R}^n$, we can definite the new tiling $T + x$ by translating every tile in T.
- If $T \neq T + x$ for all $x \in \mathbb{R}^n$, then T is **aperiodic**

Aperiodic tilings of the plane



Ways to make a tiling

We frequently start with a finite set $P = \{p_1, p_2, \ldots, p_n\}$, called prototiles.

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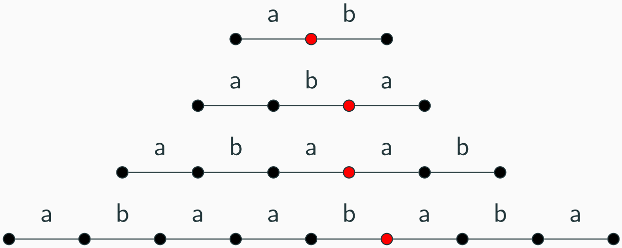
Substitutions

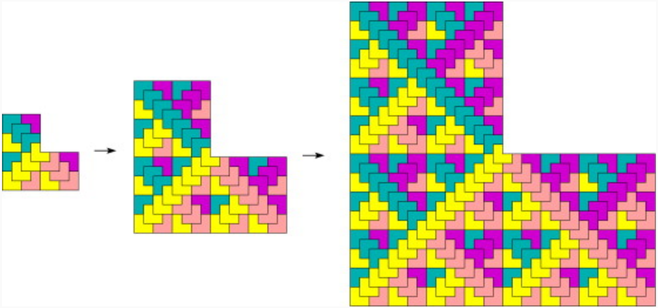
Given a prototile set, we can form a tiling by **substitution** if we have:

- A scaling constant $\lambda > 1$
- A rule ω such that, for any prototile $p \in P$, $\omega(p)$ is a patch with suppoart λP and whose tiles are translates of members of P.

Fibonacci Tiling of $\mathbb{R}$



2d example



How do we make a tiling space?

The tiling metric

The distance between two tilings T_1 and T_2 is less than ϵ if T_1 and T_2 agree on a ball around the origin, of radius less than $\frac{1}{\epsilon}$, up to translation by at most ϵ. The distance between the tilings is the infimum of these values or $\frac{1}{\sqrt{2}}$ if no such ϵ exists.

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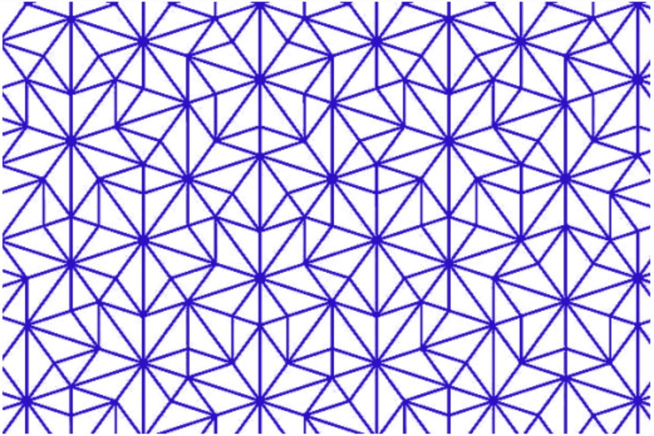
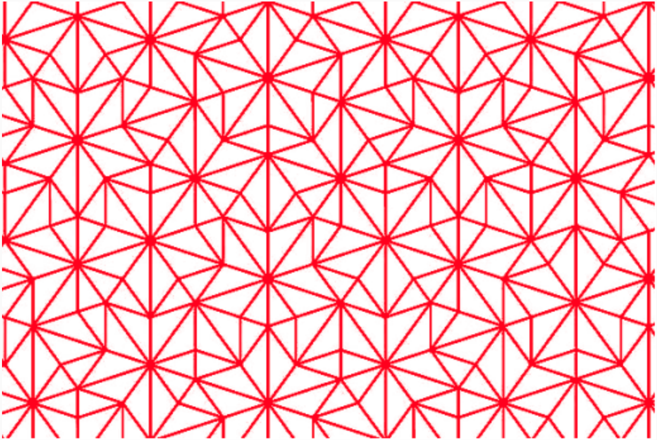
$$d(T_1, T_2) = \inf\left(\frac{1}{\sqrt{2}} \cup \{\epsilon : T_1 + u \text{ and } T_2 + v \text{ agree on } B_{\frac{1}{\epsilon}}(0), ||u||, ||v|| < \epsilon\}\right)$$

Examples of close tilings



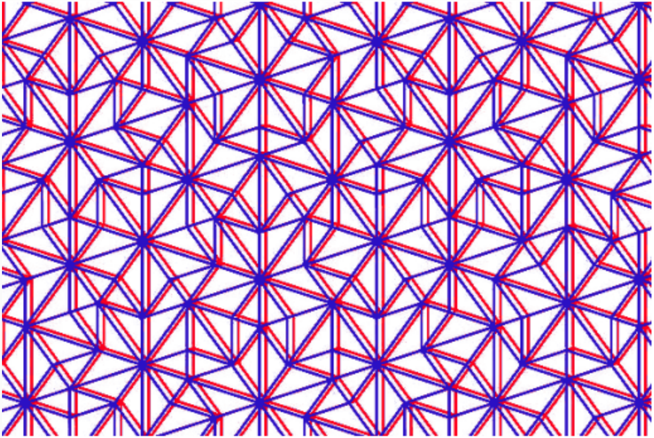
T_1

Examples of close tilings



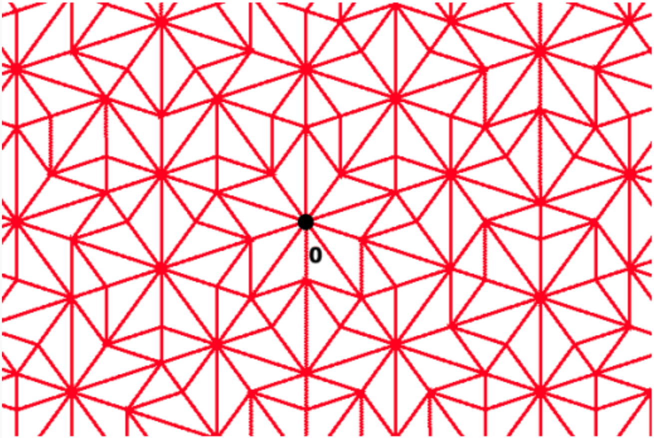
T_2

Examples of close tilings



T_1 is just a small shift of T_2

Examples of close tilings



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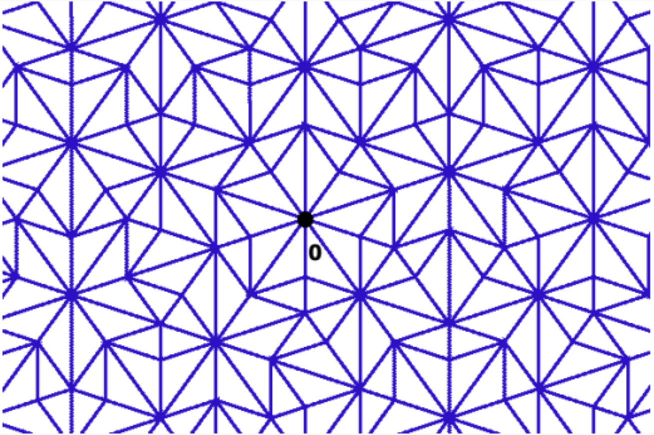

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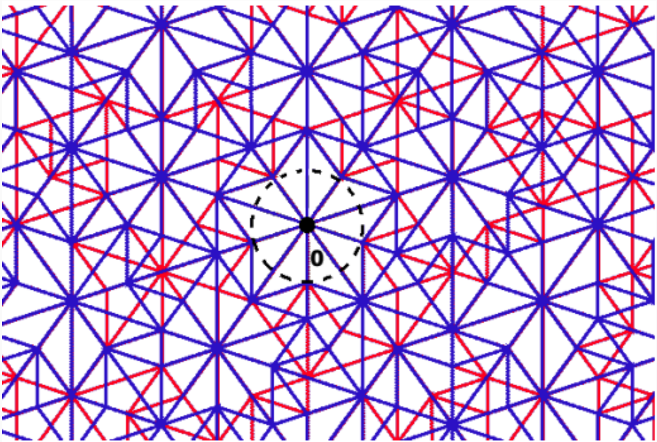

$$d(T_1, T_2) = \frac{1}{\text{radius of the ball around the origin}}$$

Defining a Tiling Space

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- It is a sequentially compact metric space
- It is also a Smale Space

Smale's Axiom A Systems

An Axiom A system is a map f on a smooth manifold M, satisfying the conditions that

- The non-wandering set of f, $\Omega(f)$ is hyperbolic and compact.
- The periodic points of f are dense in $\Omega(f)$.

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A Smale Space is an Axiom A system where

- Each point is the intersection of a locally stable set, and a locally unstable set.
- In a tiling space, the stable set is the tilings that agree with it on a large ball around the origin.
- The unstable set is the tilings that agree after small translations.

What tools can we use to study the tiling space?

Inverse Limits

- Assume we have a collection of topological spaces Γ_n and continuous maps $f_n : \Gamma_{n+1} \to \Gamma_n$.

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- The **inverse limit space** of a collection of topological spaces as above is

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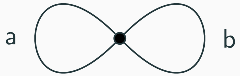
- Under suitable hypotheses, tiling spaces are inverse limit spaces
- There are theorems for dealing with inverse limit spaces!

Fibonacci Example

- Let each Γ_i be this CW-complex.



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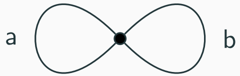

- Let f_i be the substitution map for the Fibonacci tiling, where

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- The problem here is, it looks like bb is an acceptable patch, but it never actually shows up in the tiling space.

Forcing the border

A tiling space Ω with substittion map ω **forces it's border** if, given two tilings T, T' and a point $t \in T$, $t \in T'$, there exists a positive integer N such that $\omega^N(T)$ and $\omega^N(T')$ coincide.

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Helpful Theorem!

If a substitution forces it's border, then the inverse limit of the component spaces under the substitution map is homeomorphic to the tiling space.

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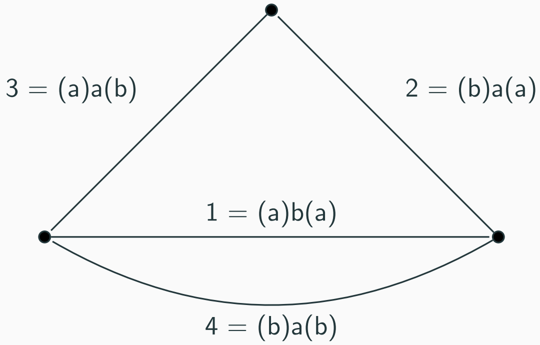

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How can we distinguish them?

Time for Algebraic Topology!

Since a tiling space is an inverse limit space, and that the Čech cohomology of an inverse limit space is isomorphic to the direct limit of the singular cohomology of the individual spaces in the inverse limit, we can actually compute the Čech cohomology of a tiling space.

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Since a tiling space is an inverse limit space, and that the Čech cohomology of an inverse limit space is isomorphic to the direct limit of the singular cohomology of the individual spaces in the inverse limit, we can actually compute the Čech cohomology of a tiling space.

$$\check{H}^n(\varprojlim(\Gamma, \varphi) \cong \varinjlim(H^n(\Gamma), \varphi^*)$$

where φ is the bonding map and φ^* is the induced map on the cohomology groups of Γ.

For a future talk!

- If the Čech cohomology groups or the two tiling spaces are different, then the tilings are combinatorially different.
- Direct limits can be calculuated in this case with linear algebra and symbolic dynamics.

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- Why does it have to be Čech cohomology? Why do homology, homotopy, and singular/simplicial cohomology fail?
- What are C^*-algebras, and why do they help in this case?
- What is Putnam homology, and why might it give us more information?