MCS 221 - Weekly Homework - Spring 2018

Jeff Ford

1. Due 2/19/18 - Driving along, Dr. Ford notices that the last four digits of his cars odometer are a palindrome. A mile later, the last five digits are a palindrome. One more mile and the middle four digits are a palindrome. One last mile, and all six digits are a palindrome. What was the odometer reading when the problem started? Assume the odometer has six digits.
   - Write out a system of equations, and a paragraph explaining how they model this problem.
   - Solve the system. You do not need to type each step, but you should include a description of how it was solved.

2. Due 2/26/18 - Prove that, if the vectors $\vec{s}$ and $\vec{t}$ both satisfy a homogeneous system of linear equations, then so do the following vectors:
   - $\vec{s} + \vec{t}$
   - $3\vec{s}$
   - $k\vec{s} + m\vec{t}$ for all $k, m \in \mathbb{R}$

   For each part, make sure you are using the definition of a vector belonging to the solution set.

3. Due 3/5/18 - Given two 2×2 non-singular matrix A and a second matrix B, which may or may not be singular. Prove that each column of B is a linear combination of the columns of A. Generalize this to the case when A and B are $n \times n$ matrices.

4. Due 3/12/18 - Prove that $|\vec{u}| = |\vec{v}|$ if an only if $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are perpendicular. You can assume that $\vec{u}$ and $\vec{v}$ have all real entries.

5. Due 3/19/18 - Suppose that A is a non-singular matrix, and B is a matrix such that the multiplication AB is well-defined. Show that $\mathcal{N}(B) = \mathcal{N}(AB)$. Note that to show that two sets are equal, you must show $\mathcal{N}(B) \subseteq \mathcal{N}(AB)$ and $\mathcal{N}(AB) \subseteq \mathcal{N}(B)$.

6. Due 3/26/18 - Given the matrix

$$A = \begin{pmatrix} 2 & -1 & 5 & -3 \\ -5 & 3 & -12 & 7 \\ 1 & 1 & 4 & -3 \end{pmatrix}.$$

   Find each of the following and prove you are correct.

   - A linearly independent subset of the columns of A which spans $\mathcal{C}(A)$.
   - A linearly independent set that spans $\mathcal{A}$, using none of the columns of A.
   - A linearly independent set that spans $\mathcal{N}(A)$.
   - A linearly independent set that spans $\mathcal{R}(A)$.

7. Due 4/9/18 - Prove the following statements about a vector space V. You can assume all scalars are in $\mathbb{C}$. Keep in mind that you know nothing about what vectors look like in V. Your proofs should be based on the definitions and properties of vector spaces. There should be no columns or matrices, and no examples.

   - If $\vec{u}, \vec{v}, \vec{w} \in V$, and $\vec{w} + \vec{u} = \vec{w} + \vec{v}$, show that $\vec{u} = \vec{v}$.
   - If $\vec{u}, \vec{v} \in V$, and α is a non-zero scalar, prove that if $\alpha\vec{u} = \alpha\vec{v}$, then $\vec{u} = \vec{v}$
   - If $\vec{u} \neq \vec{0}$ is a vector in V, with α, β, show that if $\alpha\vec{u} = \beta\vec{u}$, then $\alpha = \beta$.

8. Due 4/16/18 - We call a sequence of numbers a $F2$ sequence, if, given the first two terms, each of the terms is the sum of the two that proceeded it. For example, if the first terms are 0 and 1, the sequence is $0, 1, 1, 2, 3, 5, \ldots$, which is the usual Fibonacci sequence. However, if the first terms are $-2, -1$, the sequence would be $-2, -1, -3, -4, -7, -11, \ldots$.

Let V be the set of all $F2$ sequences, where the first two terms can be any real number. Assume you only use real numbers for coefficients, and prove that V is a vector space. You can assume that, for two sequences $(a) = (a_0, a_1, a_2, \ldots)$ and $(b) = (b_0, b_1, b_2, \ldots)$, that $(a + b) = (a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots)$ and for some $\alpha \in \mathbb{R}$, $\alpha(a) = (\alpha a_0, \alpha a_1, \alpha a_2, \ldots)$. Make sure you verify all of the properties in section VS of the textbook.

9. No weekly homework for 4/23/18

10. Due 4/30/18 - Let A be an $n \times n$ matrix, with distinct eigenvalues. Prove that the associated eigenvectors form a basis for $\mathbb{R}^n$. Construct a counterexample for a 2×2 case where the eigenvalues aren't distinct. Cite theorems from the textbook where appropriate.

11. Due 5/7/18 - Assume A is a square matrix, with eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that $\det(A) = \lambda_1 \lambda_2 \ldots \lambda_n$

12. Due 5/14/18 - Let $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$. Find the eigenvalues and eigenvectors of A. Make a matrix whose columns are the eigenvectors of A, and label it P. Let B be the diagonal matrix whose non-zero entries are the eigenvalues of A. Show that $A = PBP^{-1}$

13. Due 5/21/18 - Recall the Rank-Nullity Theorem. Let V and W be arbitrary vector spaces.

    - If a linear transformation $T : V \to W$ is onto, prove that the dimension of W must be less than or equal to the dimension of V.
    - If the dimension of W is not greater than the dimension of V, must $T : V \to W$ be a linear transformation? Prove or disprove.